Journal of Natural & Physical Sciences

Vol 9-12

1995-98

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Volume 9-10 (1995-96) —98

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

JOURNAL OF NATURAL & PHYSICAL SCIENCES

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151083

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Gurukul Kangri Vishwavidyalaya, Hardwar

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका
Journal of Natural & Physical Sciences

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ON AN ABSTRACT FUNCTIONAL INTEGRODIFFERENTIAL EQUATION

M.B. Dhakne*

(Received 07-03-95)

ABSTRACT

The aim of the present paper is to study the existence of a unique weak solution of an abstract functional integrodifferential equation of the more general type by using the well known Banach fixed point theorem and the integral inequality established by Panchpatte.

INTRODUCTION

Let X be a general Banach space. Let $C = C([-r, 0], X)$, $0 < r < \infty$, denotes the space of continuous functions mapping $[-r, 0]$ into X with norm $\|\phi\|_C = Sup_{\theta\in[-r,0],} \|\phi\theta\|$, for $\phi\theta\in C$. If x is a continuous function from $[-r,T]$, $(T>0)$ to X and $t\in[0, T]$, then x_t denotes the element of C given by $x_t(\theta) = x(t+\theta)$ for $\theta\in[-r, 0]$. In the present paper, we study the following abstract functional integrodifferential equation of the type.

$$x'(t) + A(t)x(t) = f\left(t, x_t, \int_0^t k(t,s,x_s)ds\right)$$

(1.1) $t\in[0, T]$,

$$x_0 = \phi[t], \quad -r \le t \le t\, 0,$$

where $x : [-r, T] \to X$, $A(t) : D(A(t)) \subset X \to X$, the functions $k : [0, T] \times [0, T] \times C \to X$ and $f : [0, T] \times C \times X \to X$ are continuous.

The equations of this type or its special forms commonly come across in almost all areas of applied Mathematics, see [1,2] and references listed therein. The problems of existence, uniqueness and other properties of the solution of the equation (1.1) or its special forms have been studied by many authors in the literatute by using different techniques, see [3-10, 12]. In particular, Kartsators and Parrott [9,10] have proved the existence of a unique weak solution of equation (1.1) when $k = 0$. In a recent paper [4], Dhakne and Pachpatte have investigated an approximation scheme for the special form of the equation (1.1) with X^* uniformly convex, see also [5,6].

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The main purpose of the present paper is to study the existence and stability of the weak solution of the equation (1.1) by using the well known Banach fixed point theorem and the integral inequality established by Pachpatte [11].

PRELIMINARIES AND STATEMENT OF RESULTS

Before proceeding to the statements of our main results, we shall setforth some preliminaries from [9] and hypotheses on the functions involved in (1.1) that will be used in our subsequent discussion.

An operator $A : D \subset X \to X$ is said to be m-accretive if

$$\| x - y + \lambda (Ax - Ay) \| \geq \| x - y \|$$

for every $x, y \in D$, $\lambda > 0$ and $R(I + \lambda A(t)) = X$.

By a weak solution $\bar{x}(t)$ of the equation (1.1) on $[-r, T]$, we mean for $t \in [-r, 0]$, $\bar{x}(t) = \phi(t)$, for $t \in [0, T]$, $\bar{x}(t) = U(t,0)\phi(0)$, where $\{U(t,s) : 0 \leq s \leq t \leq T\}$ is a family of evolution operators on $[0, T]$ for the problem,

$$(2.1) \qquad t \in [0, T],$$

$$x(0) = \phi(0)$$

(See Evans [8]).

We list the following hypotheses for convenience.

(H_1) $A(t)$ is m-acdretive for each $t \in [0, T]$.

(H_2) There exists a constant $\lambda_0 > 0$, a continuous nondecreasing function $r_1 : [0, \infty) \to [0, \infty)$ and a continuous function $p : [0, T] \to X$ such that for all $\lambda \in (0, \lambda_0)$, $t, s \in [0, T]$ and $x \in \overline{D(A(t))}$

$$(2.2) \qquad \|(I + \lambda A(t))^{-1} x - (I + \lambda A(s))^{-1} x\| \leq \|p(t) - p(s)\| \, r_1(\|x\|).$$

(H_3) There exist continuous functions $m : [0, T] \to [0, \infty)$, $q : [0, T] \times [0, T] \to X$ and a continuous nondecreasing function $r_2 : [0, \infty) \to [0, \infty)$ such that

$$(2.3) \qquad \|k(t,s,\phi) - k(t,s,\psi)\| \leq m(s) \|\phi - \psi\|_C$$

$$(2,4) \qquad \|k(t_1,s_1,\phi) - k(t_2,s_2,\phi)\| \leq \|q(t_1,s_1) - q(t_2,s_2)\| r_2(\|\phi\|_C)$$

for all $\phi, \psi \in C, t, t_1, t_2, s, s_1, s_2, \in [0,T]$ and

$$(2.5) \qquad \int_0^\infty m(t)\,dt < \infty$$

(H_4) There exist continuous functions $n : [0, T] \to [0, \infty)$, $h : [0, T] \to X$ and a continuous nondecreasing function $r_3 : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that

$$(2.6) \qquad \|f(t,\phi,x) - f(t,\psi,y)\| \leq n(t)[\|\phi - \psi\|_C + \|x - y\|]$$

$$(2.7) \qquad \|f(t,\phi,x) - f(t,\phi,x)\| \leq \|h(t) - h(s)\| \, r_3(\|\phi\|)_C, \|x\|)$$

for all $\phi, \psi \in C$, $t, s \in [0,T]$ $x, y \in X$ *and*

$$(2.8) \qquad \int_0^\infty n(t) e^{-\alpha t}\,dt < \infty, \alpha < 0.$$

(H_5) The initial function $\phi \in C$ *and* $\phi(0) \in \overline{D(A(0))}$

REMARK 1 : It is to be noted that the hypotheses (H_1) and (H_2) imply that the set $\overline{D(A(t))}$ is independent of t. (See [8], Lemma 3.1).

We state the following lemma established by Pachpatte in [11] which is effectively employed in the proof of our main result.

Lemma : (See pachpatte [11, p. 758]). Let $a(t)$, $b(t)$ and $c(t)$ be real valued nonnegative continuous functions defined on R^+, for which the inequality

$$c(t) \le c_0 + \int_0^t a(s)c(s)ds + \int_0^t a(s)[\int_0^s b(\tau)c(\tau)d\tau]ds,$$

holds for all $t \in R^+$, where c_0 is a nonnegative constant. Then

$$c(t) \le c_0[1 + \int_0^t a(s)\exp\{\int_0^s (a(\tau)+b(\tau))d\tau\}ds]$$

for all $t \in R^+$

Our main results are established in the following theorems.

THEOREM 1. Suppose that the hypotheses (H_1) - (H_5) hold. Then there exists a unique weak solution $\bar{x}(t)$ of the equation (1.1) on $[-r,T]$.

In the following theorem, we assume that solutions of the equation (1.1) exist on $[-r,\infty)$.

THEOREM 2. Suppose that $A(t) + \alpha I$ is m-accretive for all $t \in [0,\infty)$, where α is a fixed negative constant. Further, suppose that hypotheses (H_2) - (H_5) are fulfilled with $[0, T]$ replaced by $[0, \infty)$. If $\bar{x}(t)$ and $\bar{y}(t)$ are the weak solutions of (1.1) and

$$y'(t)+A(t)y(t)=f(t,y_t,\int_0^t k(t,s,y_s)ds),t \in [0,\infty),$$

(2.9) $$y_0 = \psi(t), \qquad -r \le t \le 0,$$

respectively on $[-r, \infty)$, then the estimate

$$\|\bar{x}(t)-\bar{y}(t)\| \le L\|\phi-\psi\|_C\, e^{\alpha t}$$

where L is suitable constant, holds for all $t \in [0,\infty)$.

Remark 2. We note that in [12], Pachpatte has established several interesting results regarding the boundedness, stability and asymptotic behaviour of solutions of the equation (1.1) without functional argument when $A(t)$, for each $t \in R^+$ is linear closed operator by using comparision principle and the integral inequality of the more general type investigated by himself in [11]. Here our conditions on functions involved in (1.1) and the approach to the problem are different.

PROOFS OF THEOREMS 1 AND 2

The nonnegative functions m and n being continuous on compact interval $[0, T]$, there exist positive constants M and N such that $m(t) \leq M$ and $n(t) \leq N$ for $t \in [0, T]$. Let $S = \{u \in C([-r, T], X);\ u(t) = \phi(t)$ for $t \in [-r, 0]\}$. For $u \in S$, define $\| u \|_B = sup_{t \in [-r, T]} e^{\beta t} \| u(t) \|$, where β is a fixed negative constant such that $\beta^2 > N(2+MT)|\beta| + 2NM$. It is apparent that S is a complete metric space with this Bielecki norm. Let $u \in S$ and define $T : S \to S$ by

$$Tu(t) = \begin{cases} \phi(t) \ for\ t \in [-r, 0], \\ x_u(t) \ for\ t \in [0, t], \end{cases}$$

where $x_u(t)$ is the unique weak continuous solution of the equation

$$x'(t) + A(t)x(t) = f(t, u_t, \int_0^t k(t, s, u_s)ds), t \in [0, T],$$

(3.1) $\qquad x(0) = \phi(0)$

on $[0, T]$. The existence of such a solution is guaranteed by Evans [8, Theorem 1]. We now, show that T is a contraction operator on S.

Consider the following equation

$$x'(t) = A(t)x(t) = f(t, v_t, \int_0^t k(t, s, v_s)ds), t \in [0, T],$$

(3.2) $\qquad x(0) = \phi(0)$

where $v \in S$. Clearly, this aquation (3.2) has the unique strongly continuous weak solution $x_v(t)$ on $[0, T]$. We note that for $t \in [-r, 0]$, $Tu(t) = Tv(t)$. By making use of [8, Theorem 3], (2.3) and (2.6) we have, for $t \in [0, T]$,

(3.3)
$$\|Tu(t) - Tv(t)\| = \|x_u(t) - x_v(t)\|$$

$$\leq \int_0^t \|f(s, u_s, \int_0^s k(s,\tau,u_\tau)d\tau) - f(s, v_s, \int_0^s k(s,\tau,v_\tau)d\tau\|ds$$

$$\leq N \int_0^t \|u_s - v_s\|_C\, ds + NM \int_0^t \int_0^s \|u_\tau - v_\tau\|_C\, d\tau ds$$

which yields,

(3.4)
$$e^{\beta t}\|x_u(t) - x_v(t)\|$$

$$\leq Ne^{\beta t} \int_0^t \|u_s - v_s\|_C\, as \ + MN e^{\beta t} \int_0^t \int_0^s \|u_\tau - v_\tau\|_C\, d\tau ds$$

$$= N \int_0^t e^{\beta(t-s)} e^{\beta s}\|u_s - v_s\|_C\, ds$$

$$+ MN \int_0^t \int_0^s e^{\beta(t-\tau)} e^{\beta\tau}\|u_\tau - v_\tau\|_C\, d\tau\ ds$$

Now, we have

(3.5)
$$e^{\beta s}\|u_s - v_s\|_C = e^{\beta s} sup_{\theta \in [-r,0]}\|u(s+\theta) - v(s+\theta)\|$$

$$= sup_{\theta \in [-r,0]}\|e^{-\beta\theta} e^{\beta(s+\theta)}[u(s+\theta) - v(s+\theta)]\|$$

$$\leq sup_{s^* \in [-r,T]}\|e^{\beta s^*}[u(s^*) - v(s^*)]\|$$

$$= \|u - v\|_B$$

using (3.3), (3.4) and (3.5), we have for $t \in [0,T]$,

$$e^{\beta t}\|Tu(t) - Tv(t)\|$$

$$\leq N \int_0^t e^{\beta(t-s)}\|u-v\|_B\, ds + NM \int_0^t \int_0^s e^{\beta(t-\tau)}\|u-v\|_B d\tau ds$$

$$\leq \frac{2N}{\|\beta\|}\|u-v\|_B + \frac{NM(2+|\beta|T)}{|\beta|^2}\|u-v\|_B$$

$$= \frac{(2N + NMT)|\beta| + 2NM}{|\beta|^2} ||u - v||_B$$

$$\leq ||u - v||_B$$

Hence, we get

$$(3.6) \qquad ||Tu - Tv||_B \quad \leq \quad ||u - v||_B$$

and consequently T is a contraction. Therefore, T has a unique fixed point, say $\bar{x}$. Then, $\bar{x}$ is the unique weak solution of the equation (1.1) on [-r,T]. This completes the proof of the Theorem 1.

Let $\bar{x}$(t) and $\bar{y}$(t) be the weak solutions of the following equations

$$x'(t) + A(t)x(t) = f(t, x_t, \int_0^t k(t, s, x_s)ds), \qquad t \geq 0,$$

$$x_0 = \phi(t), -r \leq t \leq 0$$

and
$$y'(t) + A(t)y(t) = f(t, y_t, \int_0^t k(t, s, y_s)ds), \qquad t \geq 0,$$

$$y_0 = \psi(t), -r \leq t \leq 0,$$

respectively on $[-r, \infty]$ obtained as in Theorem 1. Using the m-accretiveness of $A(t) + \alpha I$, the proofs of theorems 1 & 3 of Evans [8], (2.3) and (2.6) we get the inequality for $t \geq 0$.

$$||\bar{x}(t) - \bar{y}(t)|| \quad \leq e^{\alpha t} || \phi(0) - \psi(0) ||$$

$$+ \int_0^t e^{\alpha(t-s)} || f(s, \bar{x}_s, \int_0^s k(s, \tau, \bar{x}_\tau)d\tau) - f(s, \bar{y}_s, \int_0^s k(s, \tau, \bar{y}_\tau)d\tau)|| ds$$

$$\leq e^{\alpha t} ||\phi(0) - \psi(0)|| + \int_0 n(s)e^{\alpha(t-s)} ||\bar{x}_s - \bar{y}_s||_C \cdot ds$$

$$+ \int_0 n(s)e^{\alpha(t-s)} \int_0^s m(\tau) ||\bar{x}_\tau - \bar{y}_\tau||_C \cdot d\tau \, ds$$

$$\leq e^{\alpha t}\|\phi-\psi\|_C+e^{\alpha t}\int_0^t n(s)e^{-\alpha s}e^{-\alpha s}\|\bar{x}_s-\bar{y}_s\|_C\,ds$$

$$+e^{\alpha t}\int_0^t n(s)e^{-\alpha s}\int_0^s m(\tau)e^{-\alpha\tau}\|\bar{x}_\tau-\bar{y}_\tau\|_C\,d\tau\,ds$$

which yields

(3.7) $$e^{-\alpha t}\|\bar{x}(t)-\bar{y}(t)\|$$

$$\leq\|\phi-\psi\|_C\int_0^t n(s)e^{-\alpha s}e^{-\alpha s}\|\bar{x}_s-\bar{y}_s\|_C\,ds$$

$$+\int_0^t n(s)e^{-\alpha s}\int_0^s m(\tau)e^{-\alpha\tau}\|\bar{x}_\tau-\bar{y}_\tau\|_C\,d\tau\,ds$$

Case 1 : Suppose $t\geq r$. Then for every $\theta\in[-r,0]$, we have $t+\theta\geq 0$. For each θ's, from (3.7), we obtain

$$e^{-\alpha(t+\theta)}\|\bar{x}(t+\theta)-\bar{y}(t+\theta)\|$$

$$\leq\|\phi-\psi\|_C+\int_0^{t+\theta} n(s)e^{-\alpha s}e^{-\alpha s}\|\bar{x}_s-\bar{y}_s\|_C\,ds$$

$$+\int_0^{t+\theta} n(s)e^{-\alpha s}\int_0^s m(\tau)e^{-\alpha\tau}\|\bar{x}_\tau-\bar{y}_\tau\|_C\,d\tau ds$$

$$\leq\|\phi-\psi\|_C+\int_0^t n(s)e^{-\alpha s}e^{-\alpha s}\|\bar{x}_s-\bar{y}_s\|_C\,ds$$

$$+\int_0^t n(s)e^{-\alpha s}\int_0^s m(\tau)e^{-\alpha\tau}\|\bar{x}_\tau-\bar{y}_\tau\|_C\,d\tau ds$$

which yields

(3.8) $$e^{-\alpha t}\|\bar{x}_t-\bar{y}_t\|_C\leq\|\phi-\psi\|_C+\int_0^t n(s)e^{-\alpha s}\|\bar{x}_s-\bar{y}_s\|_C\,ds$$

$$+\int_0^t n(s)e^{-\alpha s}\int_0^s m(\tau)e^{-\alpha\tau}\|\bar{x}_\tau-\bar{y}_\tau\|_C\,d\tau\,ds$$

Case 2 : Suppose $0 \le t < r$. Then for all $\theta \in [-r, -t]$, we have $t+\theta < 0$. For such θ's, we have

$$e^{-\alpha(t+\theta)}\|\bar{x}(t+\theta) - \bar{y}(t+\theta)\|$$

$$= e^{-\alpha(t+\theta)}\|\phi(t+\theta) - \psi(t+\theta)\| \le \|\phi - \psi\|_C$$

and hence, we get,

$$(3.9) \qquad \|\bar{x}_t - \bar{y}_t\|_C \le \|\phi - \psi\|_C$$

since $\quad t < r, -\alpha < 0, \quad e^{-\alpha t} < e^{-\alpha r}$.

Therefore, from (3.9), we obtain

$$(3.10) \quad e^{-\alpha t}\|\bar{x}_t - \bar{y}_t\|_C \le e^{-\alpha r}\|\phi - \psi\|_C$$

for $\theta - \in [-t, 0], t+\theta \ge 0$. Then from (3.7) we get, as in the first case,

$$(3.11) \qquad e^{-\alpha t}\|\bar{x}_t - \bar{y}_t\|_C$$

$$\le \|\phi - \psi\|_C + \int_0^t n(s)e^{-\alpha s}e^{-\alpha s}\|\bar{x}_s - \bar{y}_s\|_C \, ds$$

$$+ \int_0^t n(s)e^{-\alpha s}\int_0^s m(\alpha)e^{-\alpha\tau}\|\bar{x}_\tau - \bar{y}_\tau\|_C \, d\tau \, ds$$

Thus, for every $t \in [0, \infty]$, we have from (3.8), (3.10), (3.11)

$$(3.12) \quad e^{-\alpha t}\|\bar{x}_t - \bar{y}_t\|_C$$

$$\le e^{-\alpha r}\|\phi - \psi\|_C + \int_0^t n(s)e^{-\alpha s}e^{-\alpha s}\|\bar{x}_s - \bar{y}_s\|_C \, ds$$

$$+ \int_0^t n(s)e^{-\alpha s}\int_0^s m(\alpha)e^{-\alpha\tau}\|\bar{x}_\tau - \bar{y}_\tau\|_C \, d\tau \, ds$$

By an application of Lema with *C(t)* $e^{-\alpha t}\|\bar{x}_t - \bar{y}_t\|_C$,we get from 3.12

(3.13) $$e^{-\alpha t}\|\overline{x_t} - \overline{y_t}\|_{C}$$

$$\leq e^{-\alpha r}\|\theta - \Psi\|\left[1 + \int_0 n(s)e^{-\alpha s}\exp\left\{\int_0^s (n(\tau)e^{-\alpha\tau} + m(\tau))d\tau\right\}ds\right]$$

From (3.13), (2.5) and (2.8), it follows that

$$\|\overline{x_t} - \overline{y_t}\|_{C} \leq \mathrm{L}\|\theta - \psi\|_{C}\, e^{\alpha t}$$

where L is suitable positive constant. Since

$$\|\overline{x}(t) - \overline{y}(t)\|_{C} \leq \|\overline{x}_t - \overline{y}_t\|_{C}$$

$$\|\overline{x}(t) - \overline{y}(t)\| \leq \mathrm{L}\|\phi - \psi\|_{C}\, e^{\alpha t}$$

Hence, the solution $\overline{x}(t)$ of (1.1) is asymptotically stable and this completes the proof of the Theorem 2.

FURTHER APPLICATIONS

In order to illustrate the applications of our main results, we consider the following equation

(4.1) $$v'(\varsigma) = H(v(\varsigma)) + F(v(\tau\varsigma)), \int_1^{\varsigma} k(v(\varsigma\eta))d\eta$$

where $H, K: X \longrightarrow X$ *and* $F: X \times X \longrightarrow X$ are nonlinear operators and λ is a constant in (0,1).

Now consider the transformation

$\varsigma = e^t$ *and set* $\tau = e^{-r}, \eta = e^s, v(\varsigma) = u(t)$ in the equation (4.1) to obtain the equation

(4.2) $$u'(t) = e^t H(u(t)) + e^t F(u(t-r), \int_0 e^s k(u(s-r))ds)$$

Assume the following conditions

1. There exists $\alpha < 0$ such that -H+α isaccretive and $-R(I - \mu H) = x$ *for* $\mu > 0$.
2. *K* is a Lipschitz continuous everywhere defined operator with

Lipschitz norm β.

3. F is a Lipschitz continuous in either arguments everywhere defined operator with Lipschitz norm.
4. There is an increasing function $J:[0,\infty) \to [0,\infty)$ such that $\|H(u)\| \leq (\|u\|)$.

By using the methods [7] and [3, Lemma 3.2], it is easy to observe from conditions (1) - (4) that the hypotheses (H_1) - (H_4) hold. Now, an application of Theorem 1 yields a unique weak solution u(t) of the equation (4.2), $u_0 = \phi$, and for any $\phi \in C$ with $\phi(0) \in \overline{D(H)}$ on *[-r, T]*. Since $T \in (0,\infty)$ can be chosen arbitrarily large, this solution *u(t)* is acutally extendable ro $[-r,\infty]$. By an application of Theorem 2, the solution *u(t)* of (4.2), $u_0 - \phi$,(where ϕ is as above), is asymptotically stabe.

ACKNOWLEDGEMENT

The author expresses his sincere gratitude to Professor B.G. Pachpatte for many helpful discussions and suggestions during the preperation of this paper.

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 13-24

CONVERGENCE OF SEQUENCES OF ITERATES OF MULTIVALUED OPERATORS#

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(Received 17-07-95)

ABSTRACT

The notion of Ishikawa iterates is extended to a pair of multivalued maps and it is shown that under certain contractive conditions the limit of the sequence of Ishikawa iterates, when it converges, is a common fixed point of the maps.

Mathematics Subject Classifications (1991) : 54H25, 47H10

INTRODUCTION & PRELIMINARIES

Rhoades [11], Hicks and Kubicek [3] and several other mathematicians (cf. [9], [10], [12]) have shown that for any operator *T*, which satisfies certain contractive conditions, the limit of the sequence of Mann iterates (see [2] and [14]), whin it converges, is a fixed point of the operator *T*. On the other hand, Naimpally and Singh [9], under general contractive conditions, found that the limit of any convergent Ishikawa sequence (see [4]) is a fixed point of the operator; while Naidu and Prasad [8] studied the convergence of Ishikawa iterates of a pair of operators in respect of certain results given in [9]. Recently Kuhfittig [6] extended Krasnoselskii's method [5] to Mann iteration procedure for multivalued operators. Motivated by above ideas. Singh [15] defined Ishikawa iteration scheme for a multivalued operator and used it to approximate its fixed points.

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The original article was not written in English. This, a page by page translation of the orignal article J. Natur. Phys. Sci. Vol. 4 (1-2) (1990), pages 187-198. is being published on the request of readers.

In the present paper, we use Ishikawa iteration scheme for a pair of multivalued operators to prove two theorems. The results obtained herein generalize the results of Singh [15]. Also, the results of Rhoades [11], Hicks and Kubicek [3], Naimpally and Singh [9], Naidu and Prasad [8] and Kuhfittig [6] can be obtained as special cases of our theorems.

Generally, we shall follow the notations used in Nadler [7] and Singh [15].

Let $(X, ||.||)$ be a normed linear space, and let $CL(X)$ be the collection of nonempty closed subsets of X. Denote by $(CL(X), H)$ the Hausdorff metric space.

The following lemma is well known (see Rus [13] for details).

Lemma. Let $A, B \in CL(X)$, and let $q > 1$. Then for every $x \in A$, there exists $y \in B$ such that $d(x,y)\ D \leq qH(A,B)$.

For a nonempty closed convex subset C of X and for $S,T : C \rightarrow CL(C)$, as discussed in [15], we shall use the following iteration scheme.

(1) $x_0 \in C,$

(2) $y_n = b_n p_n + (1-b_n)x_n,\ n \geq 0,\ p_n \in Sx_n,$

(3) $x_{n+1} = (1-a_n)x_n + a_n q_n,\ n \geq 0,$

where $q_n \in Ty_n$ is such that

(4) $||p_n - q_n|| \leq k^{-h}H(Sx_n, Ty_n),\ h, k \in (0, 1),$

(5) $0 \leq a_n, b_n \leq 1$ for all n,

(6) $\lim_n a_n > 0.$

Note that if $T : C \rightarrow C$, then condition (4) is easily satisfied as $k^{-h} > 1$.

RESULTS

Theorem 1. Let C be a nonempty closed convex subset of a normed linear space X and let $S, T : C \rightarrow CL(C)$. Suppose that the sequence $\{x_n\}$, as defined earlier, converges to a point z. If for all $x, y \in C$,

(7) $H(Sx,Ty) \le k \max\{\|x-y\|, D(x, Sx), D(y, Ty), [D(y,Sx) + D(x, Ty)]\}$,

where $k \in (0, 1)$, then $z \in Sz \cap Tz$

Proof : Since $x_n \to z$ and $\{a_n\}$ is bounded away from zero,

$$\|x_{n+1} - x_n\| = a_n \|x_n - q_n\|$$

and

$$\|q_n - z\| \le \|q_n - x_n\| + \|x_n - z\|$$

imply that $\|q_n - x_n\|$ and $\|q_n - z\|$ converge to zero. Further, from (2) and triangle inequality, we have

$$\begin{aligned}
\|x_n - y_n\| &= b_n \|x_n - p_n\| \le \|x_n - p_n\|, \\
\|x_n - p_n\| &\le \|x_n - q_n\| + \|q_n - p_n\|, \\
\|y_n - q_n\| &\le \|y_n - x_n\| + \|x_n - q_n\| \\
&\le \|x_n - p_n\| + \|x_n - q_n\| \\
&\le \|x_n - q_n\| + \|p_n - q_n\| + \|x_n - q_n\| \\
&\le 2\|x_n - q_n\| + \|p_n - q_n\|
\end{aligned}$$

and

$$\|y_n - p_n\| = (1 - b_n)\|x_n - p_n\| \le \|x_n - p_n\|.$$

Therefore

$$\begin{aligned}
\|p_n - q_n\| &\le k^{-h} H(Sx_n, Ty_n) \\
&\le k^{-h} k \max\{\|x_n - y_n\|, D(x_n, Sx_n), D(y_n, Ty_n); [D(x_n, Ty_n) \\
&\qquad + D(y_n, Sx_n)]\}
\end{aligned}$$

$$\leq \quad k' \max\{\|x_n - p_n\|, \|y_n - q_n\|, \|x_n - q_n\| + \|y_n - p_n\|\}$$

where $k' = k^{1-h}$.

Hence

$$\|p_n - q_n\| \quad \leq \quad (2k'/(1 - k'))\|x_n - q_n\|.$$

Therefore making $n \to \infty$,

$$\|p_n - q_n\| \to 0, \qquad \|x_n - p_n\| \to 0,$$

and

$$\|p_n - z\| \to 0, \qquad \|y_n - q_n\| \to 0$$

and from (2)

$$\|y_n - z\| \to 0.$$

From condition (7),

$$\begin{aligned} D(z, Sz) &\leq \|z - q_n\| + D(q_n, Sz) \\ &\leq \|z - q_n\| + H(Ty_n, Sz) \\ &\leq \|z - q_n + k \max\{\|y_n - z\|, D(y_n, Ty_n), D(z, Sz), \\ &\qquad [D(y_n, Sz) + D(z, Ty_n)]\} \\ &\leq \|z - q_n\| + k \max\{\|y_n - z\|, \|y_n - q_n\|, D(z, Sz), \\ &\qquad \|y_n - z\| + D(z, Sz) + \|z - q_n\|\}. \end{aligned}$$

Hence by making $n \to \infty$, we have $z \in Tz$.

Now for $S, T : C \rightarrow CL(C)$, let us consider the following conditions.

(8a) $H(x, Sx) + H(y, Ty) \leq a||x-y||,\ 1 \leq a < 2,$

(8b) $H(x, Sx) + H(y, Ty) \leq b[D(x,Ty) + D(y, Sx) + ||x-y||],\ 1/2 \leq b < 2/3,$

(8c) $H(x, Sx)+H(y,Ty)+H(Sx, Ty) \leq c[D(x,Ty)+D(y, Sx),1 \leq c < 3/2,$

(8d) $H(Sx, Ty) \leq k \max \{||x-y||, Dx, Sx), D(y,Ty),$

$$1/2\ [D(x, Ty)+D(y,Sx)],\ 0 \leq k < 1.$$

Theorem 2. Let C be a nonempty closed convex subset of a normed linear space X, and let $\{x_n\}$ be the sequence as defined earlier converging to a point z. If for every $x, y \in C$, S and T satisfy at least one of the conditions (8a) - (8d) and,

(8e) $k^{-h} \leq 2$ or $\lim_n b_n > (c-1-k^h)/c+1),$

then $z \in Sz \cap Tz$.

Proof. We shall repeatedly use the following statement due to Ciric [1]. For any $x \in C$ and $y \in B$, where $B \in CL\ (C)$, we have $||x-y|| \leq H(x, B)$.

Since $p_n \in Sx_n$ and $q_n \in Ty_n$, we have

$$\begin{aligned} 2||p_n - q_n|| &\leq ||x_n - p_n|| + ||y_n - q_n|| + ||x_n - q_n|| + ||y_n - p_n|| \\ &\leq H(x_n, Sx_n) + H(y_n, Ty_n) + ||x_n - q_n|| + ||y_n - p_n||. \end{aligned}$$

Now, as in the case of Theorem 1, it can be easily varified that

$$\|x_n - q_n\| \to O_x \|q_n - z\| \to 0.$$

Setting $x = x_n$ and $y = y_n$ in (8a), we get

$$\begin{aligned} 2\|p_n - q_n\| &\leq a\|x_n - y_n\| + \|x_n - q_n\| + \|y_n - p_n\| \\ &\leq a\, b_n (\|x_n - q_n\| + \|q_n - p_n\|) + \|x_n - q_n\| \\ &\quad + (1 - b_n)(\|x_n - q_n\| + \|q_n - p_n\|) . \end{aligned}$$

Hence

$$\|p_n - q_n\| \leq t_1 \|x_n - q_n\|,$$

where $t_1 = [2+(a-1)b_n]/[1-(a-1)b_n]$.

Again setting $x = x_n$ and $y = y_n$ in (8b), we have

$$\begin{aligned} 2\|p_n - q_n\| &\leq b(D(x_n, Ty_n) + D(y_n, Sx_n) + \|x_n - y_n\| + \\ &\quad (\|x_n - q_n\| + \|y_n - p_n\|)) \\ &\leq (b + 1)(\|x_n - q_n\| + \|y_n - p_n\|) + b\|x_n - y_n\| \end{aligned}$$

that is

$$\|p_n - q_n\| \leq t_2 \|x_n - q_n\|,$$

where $t_2 = (2 + 2b - b_n)/(1 - b + b_n)$.

If $x = x_n$ and $y = y_n$ in (8c), then

$$||p_n - q_n|| \leq t\, H(Sx_n, Ty_n), \text{ where } t = k^{-h}.$$

Therefore

$$\begin{aligned}
(2t + 1)||p_n - q_n|| &= t||p_n - q_n|| + t||p_n - q_n|| + ||p_n - q_n|| \\
&\leq t(||p_n - y_n|| + ||y_n - q_n||) + t(||x_n - p_n|| + ||x_n - q_n||) \\
&\qquad + t\, H(Sx_n, Ty_n) \\
&\leq t\, H(y_n, Ty_n) + t\, H(x_n, Sx_n) + t\, H(Sx_n, Ty_n) \\
&\qquad + t(||x_n - y_n|| + ||y_n - p_n||) \\
&\leq ct\, D(x_n - Ty_n) + D(y_n, Sx_n) + t(||x_n - q_n|| + ||y_n - p_n||) \\
&\leq t(c + 1)(||x_n - q_n|| + ||y_n - p_n||)
\end{aligned}$$

Thus $||p_n - q_n|| \leq t_3||x_n - q_n||$,

where $t_3 = [t(c + 1)(2 - b_n)]/[1 + t(1 - c(1 - b_n) + b_n]$

Finally, if we set $x = x_n$ and $y = y_n$ in (8d), then

$$\begin{aligned}
||p_n - q_n|| &\leq tk \max\{||x_n - y_n||, D(x_n, Sx_n), D(y_n, Ty_n), \\
&\qquad 1/2[D(x_n, Ty_n) + D(y_n, Sx_n)\} \\
&\leq tk(2||x_n - q_n|| + ||q_n - p_n||).
\end{aligned}$$

Therefore

$$||p_n - q_n|| \leq t_4 \, ||x_n - q_n||,$$

where $t_4 = 2tk/(1-tk)$.

Hence for $x = x_n$ and $y = y_n$ we have

(9) $$||p_n - q_n|| \leq \max \{t_1, t_2, t_3, t_4\} \, ||x_n - q_n||.$$

Clearly, for any $b_n \in [0, 1]$, $t_4 > 0$ and $t_1, t_2 > 0$. Now keeping in view t_3, for any $b_n \in [0, 1]$, $t_3 > 0$ if $1 + t[1-c(1-b_n) + b_n] > 0$.

That is $1 + t(1-3(1 - b_n)/2 + b_n)$(since c and 3/2 may be sufficiently close).

Therefore $1 + t(1-3/2) > 0$ or $t < 2$.

It is clear that for any $b_n \in [0, 1]$, when $t = 2$, $t_3 > 0$. If $t > 2$ then $t_3 > 0$ provided

$$1 + t[1-c(1-b_n) + b_n] > 0, \text{ that is if } b_n > (c-1-1/t)/(c + 1).$$

Hence keeping in view (8e), $t_3 > 0$. Therefore from (9), it follows that

$$||p_n - q_n|| \to 0, \quad ||x_n - p_n|| \to 0,$$

and

$$||p_n - z|| \to 0 \text{ as } n \to \infty.$$

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Now we shall show that z is a fixed point of S. Hence by setting

$x = z$ and $y = y_n$ in (8a), we have

$$\text{(10a)} \quad H(s\,,\,Sz) + H(y_n\,,\,Ty_n) \leq \; a||y_n - z||\,.$$

Similarly, from (8b),

$$||y - q_n|| + H(z,\,Sz) \quad \leq \; b(D(y_n\,,\,Sz) + D(s,\,Ty_n) + \; ||y_n - z||$$

$$\leq \; b(||y_n - z|| + D(z,\,Sz) + \; ||z - q_n|| + ||y_n - z||)$$

$$\leq \; b(2||y_n - z|| + ||z - q_n|| + H(z,\,Sz)).$$

Therefore

$$\text{(10b)} \quad (1\text{-}b)H(z,\,Sz) \leq 2b(||y_n - z|| - (1 - b)||z - q_n||.$$

Again from (8c),

$$2H(\,z,\,Sz) \quad \leq \; H(\,z,\,Sz) + ||z - y_n|| + H(y_n,\,Ty_n) + H(Ty_n,\,Sz)$$

$$\leq \; ||z - y_n|| + c(D(y_n,Sz) + D(s.Sz) + ||z - q_n||)$$

$$\leq \; (c + 1)\,||z - y_n|| + cH(z,Sz) + c||z - q_n||\,.$$

Thus

$$(10c)\quad (2-c)H(z,Sz) \leq (c+1)||z - y_n|| + c||z - q_n||.$$

Finally, from (8d),

$$\begin{aligned} D(s,Sz) &\leq ||z - q_n|| + H(Ty_n, Sz) \\ &\leq ||z - q_n|| + k\max\{||y_n - z||, D(y_n, Ty_n), D(z,Sz), \\ &\qquad 1/2[D(y_n, Sz) + D(z, Ty_n)]\}. \end{aligned}$$

Hence

$$\begin{aligned} (10d)\quad D(z, Sz) \leq ||z - q_n||\, k\max\{&||y_n - z||, ||y_n - q_n||, D(z,Sz), \\ &1/2[||y_n - z|| + D(z, Sz) + ||z - q_n||]\}. \end{aligned}$$

Hence making $n \to \infty$ in (10a) - (10d), we have

$H(z,Sz) = 0$ and $D(z, Sz) = 0$, and $z \in Sz$.

Similarly, we can prove that $z \in Tz$.

Remark 1. If we set $S = T$ in theorems 1 and 2, we get results of Singh [15].

Remark 2. If we take $S, T : C \to C$ in Theorem 1, we get Theorem 1.2 of Naimpally and Singh [9].

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 25-42

ON FAN'S BEST APPROXIMATION AND FIXED POINT THEOREMS

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(Received 26-07-95)

ABSTRACT

In this paper an up-to date development of Ky Fan's best approximation theorem is given for single valued function as well as multivalued case. A few fixed point theorems are derived as corollaries.

Mathematics Subject Classifications : Primary 47 H10, Secondary 54 H25

Consider a function $F : C \to X$, where C is a nonempty subset of a normed linear space X. We seek a point $x \in C$ which is a best approximation for fx, i.e., the problem of existence of an $x \in C$ such that

$$||x - fx|| = d(fx, C) = inf\{||fx - y|| : y \in C\} \qquad (*)$$

We note that y is a solution of (*) if and only if y is a fixed point of $P_C of$, where P_C is the metric projection on C.

If f satisfies a suitable boundary condition, for example, $fx \in C$ for all $x \in C$, i.e. $f : C \to C$, then the set of solutions of (*) coincides with the fixed point set of f (see Park [11] for details).

In the study of fixed points we note that there are different theorems dealing with different boundary conditions. We give below a list of a few such conditions.

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1. Let $f : C \to X$ with $f(\partial C) \subseteq C$, ($\partial C$ stands for the boundary of C).
2. Let $f : C \to X$ and $x \neq fx$, then the line segment $[x, fx]$ has at least two points of C.
3. If $f : C \to X$ and $x \neq fx$ then there exists a number λ (real or complex depending on whether the vector space is real or complex) such that $|\lambda| < 1$ and $y = \lambda x + (1 - \lambda)fx \in C$.
4. If $f : C \to X$ and $x \neq fx$ for $x \in \partial C$ then there exists a $y \in C$ such that $||fx - y|| < ||x - fx||$.
5. If $f : C \to X$ and $x \neq fx$ for $x \in \partial C$ then

$$\lim_{h \to 0^+} \frac{d((1-h)x + hfx, C)}{h} < ||x - fx||.$$

In case C is a ball B of radius r and center 0, the boundary conditions are:

(i) If $f : B \to X$ and $fx = \alpha x$ for some $x \in \partial B$ *then* $\alpha \leq 1$ (Leray-Schauder condition).

(ii) If $f : B \to X$ then $||fx - x||^2 \geq ||fx||^2 - ||x||^2$ *for all* $x \in \partial B$ (Altman's Condition).

A similar set of boundary conditions, for dealing with the fixed point theorems of multifunctions, were considered.

The following well-known theorem due to Ky Fan[5] yields fixed point theorems under a set of different boundary conditions.

Theorem 1 : Let C be a nonempty compact, convex subset of a normed linear space X and $F : C \to X$ a continuous function. Then there is an $x_0 \in C$ such that

$$||x_0 - fx_0|| = d(fx_0, C) = \inf\{||fx_0 - z|| : z \in C\}. \qquad (K)$$

(a) If additionally, $f(\partial C) \subseteq C$ is satisfied, then f has a fixed point.

(b) If additionally , for $x_0 \in \partial C$ with $x_0 \neq fx_0$ there exists a $y \in \overline{I_C(x_0)}$ such that $||x_0 - fx_0|| > ||y - fx_0||$, is satisfied then f has a fixed point.

(c) If additionally, for each $x_0 \in \partial C$ with $x_0 \neq fx_0$ the line segment $[x_0 , fx_0]$ contains at least two points of C, is satisfied then f has a fixed point.

Proof. We consider case (c) and show that x_0 is a fixed point of f. By Theorem 1 there exists an $x_0 \in C$ such that

$$||x_0 - fx_0|| = d(fx_0, C).$$

Take $x_0 \neq fx_0$ and let $y = \lambda x_0 + (1 - \lambda) fx_0 \in C$, *with* $0 \leq \lambda < 1$. Then

$$||x_0 - fx_0|| \leq ||y - fx_0|| = ||\lambda(x_0 - fx_0)|| < ||x_0 - fx_0||$$

since $\lambda < 1$, a contradiction. Hence $x_0 = fx_0$.

The compactness condition can not be dropped as is illustrated by the following example in a Hilbert space.

Let B be a unit ball in the Hilbert space l^2. Let f be defined on B by

$$f(x) = \{\sqrt{1-||x||^2}, x_1, x_2, \ldots, x_n \ldots\}.$$

Then $f: B \to B$ and $||fx|| = 1$. If (K) holds, then f must have a fixed point, i.e., $fx = x$ *so* $||x|| = 1$. This gives that

$$fx = x = \{0, x_1, x_2, \ldots, x_n, \ldots\} = \{x_1, x_2, x_3, \ldots, x_n, \ldots\},$$

i.e. $x_1 = 0 = x_2 = x_3 \ldots$, so $x = 0$, contradiction to $||x|| = 1$. So (K) does not hold true.

Now, results are given for approximatively compact sets. We need the following definitions (see Cheney [4]).

Let X be a normed linear space and C be a nonempty subset of X. Let $x \in X$. An element $y \in C$ is called an *element of best approximation* (or a *nearest point*) to x if

$$||x-y|| = d(x, C) = \inf \{||x-z|| : z \in C\}.$$

The set of best approximations to x is given by

$$P(x) = \{z \in C : ||x-z|| = d(x,C)\}.$$

The mapping $P : X \to C$ is called the *metric projection* of X onto C.

If $P(x) \neq \phi$ for every $x \in X$, then C is called *proximinal.* If $P(x)$ contains at most one element for every $x \in X$, then C is called a Chebyshev set.

For each x, $P(x)$ is a closed and bounded set, and is convex if C is convex. If C is a Chebyshev set, then P is a single-valued mapping of X onto C and is called *Chebyshev map* or *best approximation operator*. If the space is a Hilbert space, then it is called the *proximity map*. If C is a closed, convex subset of Hilbert space H, then C is a Chebyshev set and P is a nonexpansive map.

A sequence $\{y_n\}$ in C, where C is a subset of normed linear space X is called a *minimizing sequence* for $x \in X$ if $d(x,y_n)$ converges to $d(x,C)$.

A set C in X is said to be *approximatively compact* if for any $x \in X$ each *minimizing sequence* for x has a subsequence converging to an element of C.

If C is approximatively compact set then (i) C is proximinal, (ii) C is closed, (iii) Px is compact, (iv) if C is convex then Px is convex, (v) $P(A) = \cup\{Px : x \in A\}$ is compact for compact subset A in X.

A metric projection onto an approximatively compact set is upper semicontinuous (usc) (see Reich [15]).

A compact set is approximatively compact. However, the converse is not true. For example, the closed unit ball of an infinite dimensional uniformaly convex Banach space is approximatively compact but not compact.

The following is due to Reich [15] for approximatively compact sets.

Theorem 2: Let C be an approximatively compact, convex subset of a Banach space X and $f : C \to X$ continuous with $f(C)$ compact. Then there exists a $y \in C$ such that

$$||y - fy|| = d(fy, C).$$

Proof: Let Q be the metric projection on C. Define $F(x) = Q(f(x))$ for each $x \in C$.

Then F is usc and $F(x)$ is nonempty compact convex subset of C for each $x \in C$. Since $f(C)$ is relatively compact, so $F(C)$ is also relatively compact because the image of a compact set under an usc map with compact point images is compact. So the result follows from the Himmelberg [6].

Singh and Watson [19] proved the following result in Hilbert space for nonexpansive mappings.

Theorem 3: Let H be a Hilbert space and C is a closed, convex subset of H. Let $f : C \to H$ be a nonexpansive map with $f(C)$ bounded. Then there exists a $y \in C$ such that $||y - fy|| = d(fy, C)$.

Proof: Let $P : H \to C$ be the proximity map and $f : C \to H$. Then $P \circ f : C \to C$ is nonexpansive. Set $B = \overline{co(Pf(C))}$.

Then B is closed, bounded, and convex and $T = Pf : B \to B$. By Browder Fixed Point Theorem T has a fixed point; that is, $Pfx_0 = x_0$ for some $x_0 \in B$. Therefore, $||x_0 - fx_0|| = d(fx_0, C)$.

We give the following :

Corollary 4: If C is a closed, bounded, convex subset of H and $f : C \to H$. is a nonexpansive map then there is a $y_0 \in C$ such that

$$||y_0 - fy_0|| = d(fy_0, C).$$

The following fixed point theorems are derived.

1. Let B_r be a closed ball of radius r and centre 0 in Hilbert space H. Let $f: B_r \to H$ ba a nonexpansive map with the property that if $fx = \alpha x$ for some $x \in \partial B_r$ then $\alpha \leq 1$. Then f has a fixed point.
2. Let C be a closed convex subset of Hilbert space H and $f: C \to H$ a nonexpansive mapping. Let $f(C)$ be bounded and $f(\partial C) \subset C$. Then f has a fixed point [19].

Browder raised the following question.

Let C be a closed convex subset of Banach space X, and let $f: C \to C$ be a nonexpansive mapping . For any $k \in [0, 1)$ and any $x_0 \in C$, the mapping defined by

$$f_k(x) = kf(x) + (1 - k)x_0$$

maps C into itself and is a contraction with Lipschitz constant k. For k sufficiently close to 1, f_k is a contractive approximation of f.

By the Banach contraction principle there exists a unique fixed point x_k of f_k in C for any $k \in [0, 1)$; that is,

$$x_k = f_k x_k = kfx_k + (1-k)x_0 .$$

It is natural to ask if the sequence x_k converges to a fixed point of f. One cannot expect, in general, an affirmative answer to this question as there are nonexpansive mappings which do not have fixed points.

An affirmative answer in the following setting was given by Browder [1] and is stated next.

Theorem 5: Let C be a closed, bounded, convex subset of Hilbert space H and let $f: C \to H$ be a nonexpansive mapping. Define $f_k(x) = kf(x) + (1 - k)x_0$, where $0 < k < 1$ and x_0 is an arbitrary point in C. Let $f_k x_k = x_k$. Then x_k converges to y_0, where y_0 is a fixed point of f closest to x_0.

The following result was established by Singh and Watson [9] where f is not necessarily a self-map and C is not bounded.

Theorem 6: Let H be a real Hilbert space and C is a closed, convex subset of H. Let $f: C \to H$ be a nonexpansive mapping with $f(C)$ bounded and $f(\partial C) \subseteq C$. Suppose that $0 \in C$. Let $f_k(x) = kf(x) + (1 - k)x_0$ for some $x_0 \in C$ and $0 < k < 1$, $k \to 1$, and let $f_k x_k = x_k$. Then x_k converges strongly to y_0, where y_0 is the fixed point of f closest to x_0.

Proof: The fixed point set, $F(f)$, of f is nonempty [1] and $F(f)$ is closed and convex [1]. So there exists a unique closest point to x_0, say $y_0 = fy_0$. For the sake of convenience we take $x_0 = 0$. Now,

$$||x_k/k - y_0||^2 = ||fx_k - y_0||^2 = ||fx_k - fy_0||^2 \le ||x_k - y_0||^2;$$

that is,

$$||x_k||^2 + k^2||y_0||^2 - 2k(x_k, y_0) \le k^2(||x_k||^2 + ||y_0||^2 - 2(x_k, y_0)).$$

This gives

$$||x_k||^2 \le \frac{2k}{1+k}(x_k, y_0) \le (x_k, y_0).$$

So

$$||x_k|| \le \left(\frac{x_k}{||x_k||}, y_0\right) \le \left\|\frac{x_k}{||x_k||}\right\| ||y_0||,$$

that is $||x_k|| \le ||y_0||$.

Since $\{x_k\}$ is bounded, x_{k_i} is a subsequence of $\{x_k\}$, converges weakly to x, that is $x_{k_i} = x_i \to x$, Then

$$\begin{aligned} ||x_i - fx_i|| &= ||k_i fx_i - fx_i|| \\ &= (k_i - 1)\, ||fx_i|| \to 0 \text{ as } i \to \infty. \end{aligned}$$

Hence $x_i - fx_i \to 0$.

Recall that a mapping f is demiclosed if $x_n \rightharpoonup x$ and $fx_n \to y$ and imply that $y = fx$ ($\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence.)

Browder [1] proved that if $f : C \to H$ is a nonexpansive mapping, then $I - f$ is **demiclosed**. Since $I - f$ is demiclosed, we get $(I - f)x = 0$, that is, $fx = x$. Since $||x_i|| \leq ||y_0||$ so $||x|| \leq ||y_0||$. But y_0 is closest to $x_0 = 0$, so $||x|| = ||y_0||$. This implies that $x = y_0$, and so $x_i \rightharpoonup y_0$. Again

$$||y_0||^2 \geq ||x_i||^2 = ||x_i - y_0 + y_0||^2$$
$$= ||x_i - y_0||^2 + ||y_0||^2 + 2(x_i - y_0\,, y_0).$$

The last part of the right side goes to 0 as $i \to \infty$, giving us $||x_i - y_0|| \to 0$.

Thus x_i converges to y_0 strongly.

Since x_i is any subsequence of x_k, the sequence x_k converges strongly to y.

Remark: In this proof, the condition that $I - f$ is monotone, is not used. The following examples are given.

Example 1: Let f: $[-1\,, 1] \to R$ be given by $f(x) = 1 - x$. Then $f(-1) = 2 \notin [-1\,, 1]$, that is , f is not a self-map. But f does have a fixed point.

Example 2: Let f: $[0, \infty) \to R$ be given by $f(x) = 1/(1 + x)$. Then f is a nonexpansive mapping and $f(x_0) = x_0 = \left(\sqrt{5}-1\right)/2$. But $[0, \infty)$ is not bounded.

Definition 7: Let $M \subseteq E$ be a convex subset. A mapping $g : M \to E$ is almost affine if and only if, for all $x, y \in M$ and $u \in E$, we have

$$||g(\lambda x + (1 - \lambda)y) - u|| \leq \lambda\, ||gx - u|| + (1 - \lambda)||gy - u||$$

for each λ, with $0 < \lambda < 1$.

Recently Prolla [14] proved the following:

Theorem 8: Let M be a nonempty, compact, convex subset of a normed linear space E and $g : M \to M$ a continuous, almost affine, onto mapping. Then, for each continuous mapping $f : M \to E$ there exists an $x \in M$ satisfying

$$||gx - fx|| = d(fx, M).$$

In case $g = I$, an identity function, we get Theorem 1.

Sehgal and Singh [18] proved this result under relaxed conditions, where an approximatively compact set is used in place of compact set C.

The following lemma is useful in the proof of the result.

Lemma 9: Let M be a non empty, approximatively compact convex subset of E and let $Q : E \to 2^M$ be the metric projection of E onto M defined by

$$Q(x) = \{y \in M : ||y - x|| = d(x, M)\}.$$

Then

(a) $Q(x) \neq \phi$ and

(b) $Q(A) = \cup\{Q(x) : x \in A\}$ is compact for any compact subset A in E.

The following result is given in [18].

Theorem 10: Let M be a nonempty, convex subset of a normed linear space E and let $Q : E \to 2^M$ be the metric projection satisfying

(a) *$Q(x) \neq \phi$ for each $x \in E$, and*

(b) Q maps compact subsets of E onto compact subsets of M.

Suppose $g : M \to M$ is continuous, almost affine, onto map satisfying for any compact set $D \subseteq M, g^{-1}(D)$ is compact.

Then for any continuous mapping $f : M \to E$ with $f(M)$ relatively compact, there exists an $x \in M$ such that

$$||gx - fx|| = d(fx, M).$$

Remarks 1:

1. It may be noted that if M is an approximatively compact set, then Q satisfies conditions (*a*) and (*b*) of Theorem 10.
2. If M is a compact subset of E, then clearly $F(M)$ is compact. Further, the continuity of g implies that for any compact set D, $g^{-1}(D)$ being closed is a compact subset of M. Thus Theorem 8 is a special case of Theorem 10.
3. In case $g = I$, an identity function, and M is compact, then Theorem 1 is obtained as a corollary.
4. In case $g = I$ and M is an approximatively compact subset of E, one gets Theorem 2.

The theorem dut to Carbone and Conti [2], that extends and unifies results in this area, is given below.

Theorem 11: Let X be a Banach space and C an approximatively compact, convex subset of X. Let $f : C \to X$ be a continuous map with $f(C)$ relatively compact. Let $g : C \to C$ be continuous, onto, proper map such that $g^{-1}(z)$ is an acyclic subset of C for every $z \in C$. Then there exists an $x \in C$ such that

$$||gx - fx|| = d(fx, C).$$

Recently, extending the result of Prolla [14], Carbone [3] proved the following.

Theorem 12: Let C be a nonempty convex subset of a normed linear space X, $f : C \to X$ a continuous map and $g : C \to C$ a continuous, onto and satisfy

$$||g(\lambda x_1 + (1-\lambda)x_2) - y|| \leq \max\{||gx_1 - y||, ||gx_2 - y||\}, \text{(almost quasiconvex)}.$$

Let D be a nonempty compact subset of C and B a nonempty subset contained in a compact convex subset of C such that for each $y \in C/D$, there exists an $x \in B$ such that $u = gx \in C$ and

$$||gy - y|| \geq ||u - fy||.$$

Then there is a $z \in D$ such that

$$||gz - fz|| \leq ||y - fz|| \text{ for all } y \in C.$$

Further extension of this result is given below [12].

Recall that for $x \in C$ the inward set of C at x $I_C(x)$ is defined by $I_C(x) = \{y \in X : y = x + r(u - x),\ r > 0,\ u \in C\}$ and $\overline{I_C(x)}$ denotes the closure.

Theorem 13 : Let C be a nonempty convex subset of a normed linear space X, $f : C \to X$ a continuous map, and $g : C \to C$ a continuous, almost quasiconvex surjection. Let D be a nonempty compact subset of C and B a nonempty subset contained in a compact convex subset of C such that, for each $y \in C\backslash D$, there exists a point $x \in B$ such that $y = gx \in C$ and

$$||gy - fy|| > ||y - fu||.$$

Then there is a point $w \in D$ such that

$$||gw - fw|| \leq ||z - fw||$$

for all $z \in \overline{I_C(gw)}$.

More precisely, either

1. f and g have a coincidence point $w \in D$; or

2. there is a $w \in D$ such that $gw \in \delta C$ (boundary of C) and

$$0 < \| gw - fw \| \leq \| z - fw \| \text{ for all } z \in \overline{I_C(gw)}.$$

Now we discuss Ky Fan type results for multivalued mappings.

Let X and Y be normed linear spaces and 2^Y denote the family of nonempty subsets of Y. The mapping $F : X \to 2^Y$ is *upper semicontinuous* (*usc*) if and only if $F^{-1}(B) = \{x \in X : Fx \cap B \neq \phi\}$ is closed for each closed subset B of Y. Equivalently, F is *upper semicontinuous* (*usc*) if and only if for each $x \in X$ and open set U of Y with $Fx \subseteq U$ there exists a neighbourhood V of X with $Fz \subseteq U$ for each $z \in V \cap X$. F is lower semicontinuous (lsc) iff $F^{-1}(A) = \{x \in X : Fx \cap A \neq \phi\}$ is open for each open set A in Y. If F is both usc and lsc then F is said to be continuous.

We list some known fixed point theorems for multivalued mappings. The first is a result of Ky Fan's that extends the result of Kakutani to infinite dimensional space.

Theorem 14: Let C be a compact, convex subset of normed space X and $F : C \to 2^C$ an usc multifunction with Fx nonempty, closed, convex for all $x \in C$. Then F has a fixed point [5]

The following is due to Himmelberg [6].

Theorem 15: Let C be a nonempty convex subset of a normed linear space X and $F : C \to 2^C$ an usc multifucntion with Fx nonempty, closed, convex for all $x \in C$. *If $F(C)$* is contained in a compact subset of C then F has a fixed point.

Several interesting important results on fixed point theorems for multivalued mappings have been given by Reich [16]. He extended Ky Fan's result and then derives fixed point theorems. Reich [16] proved the following.

Theorem 16: Let C be a compact, convex subset of a locally convex space X and $F : C \rightarrow 2^X$ continuous, point-compact, point-convex map. Then there exists a $y \in C$ such that

$$d(y, Fy) = d(Fy, C). \qquad (K')$$

Recall that

$$d(A,B) = \inf \{\|a - b\| : a \in A \text{ and } b \in B\}.$$

The following is due to Sehgal and Singh [17].

Theorem 17: Let E be a locally convex Hausdorff topological vector space and K a nonempty approximatively p-compact convex subset of E. If $F : K \rightarrow E$ is a continuous multifunction with nonempty closed convex values and $F(K)$ is relatively compact, then there exists an $x \in K$ such that

$$d_p(x, F(x)) = d_p(F(x), K).$$

The following simple example is due to Waters [21] and shows that even in the specisl case of uniformly convex Banach space E, continuity of F cannot be replaced by upper semicontinuity alone.

Example: Let $E = R^2$ with the euclidean norm and let $K = [0, 1] \times \{0\}$ Clearly K is a comvex and compact.

Define $F : K \rightarrow 2^E$ by

$$F(a,0) = \begin{cases} (0,1), \text{ if } a \neq 0 \\ L = \text{the line segment}[(0,1), (1,0)], \text{if } a = 0 \end{cases}$$

F is a convex and compact valued multifunction. Then for any $A \subseteq E$,

$$F^{-1}(A) = \begin{cases} \phi, & \text{if } A \cap L = \phi \\ K, & \text{if } (0,1) \in A \\ (0,0), & \text{if } (0,1) \notin A,\ A \cap L \neq \phi. \end{cases}$$

Thus, F is an usc but not an lsc multifunction and FK is compact.

However, for any $(a, 0)$,

$$d((a,0), F(a,0)) > 1 = d(F(a,0), K) \text{ if } a \neq 0,$$

$$d((0,0), F(0,0)) = \frac{\sqrt{2}}{2} \neq d(F(0,0), K) = 0 \text{ if } a = 0.$$

Thus, the conclusion of Theorem 17 is not satisfied.

A convex space C is nonempty convex set in a vector space with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called polytopes [10]. Here X and Y are topologicas spaces.

A set $K \subset X$ is called σ-compact if K is the countable union of compact sets. A nonempty topological space is acyclic if all of its reduced Cech homology groups over rationals vanish.

For a given class **L** of multifunctions denote

$$\mathbf{L}(X,Y) = \{T : X \to 2^Y \mid T \in \mathbf{L}\};$$

$$\mathbf{L}_C = \{T = T_m T_{m-1}, \ldots, T_1 \mid T_i \in \mathbf{L}\}.$$

Using the above notation, we have the following definitions.

(1) We say that F is a Kakutani map, and write $F \in \mathbf{K}(X,Y)$, if Y is a convex space and F is usc with nonempty, compact, convex values.

(2) F is an acyclic map, written $\mathbf{F} \in \mathbf{V}(X,Y)$ if F is usc with compact acyclic values.

(3) $F \in \mathbf{K}^+(X,Y)$ (resp. $\mathbf{V}^+(X,Y)$) if, for any σ-compact K for X there is a $\Gamma \in \mathbf{K}(K,Y)(\mathbf{V}(K,Y))$ such that $\Gamma x \subset Fx$ for each $x \in K$.

(4) $F \in \mathbf{K}^+(X,Y)$ (resp. $\mathbf{V}^+(X,Y)$) if, for any σ-compact K for X there is a $\Gamma \in \mathbf{K}_C(K,Y)(\mathbf{V}_C(K,Y))$ such that $\Gamma x \subset Fx$ for each $x \in K$.

It is known that $\mathbf{K}_C{}^+$ contains $\mathbf{K}$ and $\mathbf{K}_C$. Moreover, it is clear that $\mathbf{V}_C{}^+$ includes $\mathbf{V}_C$ and $\mathbf{K}_C{}^+$.

Lemma 18: Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E and $F \in \mathbf{V}_C{}^+(X,X)$. If F is a compact, then F has a ficed point [13].

Theorem 19: Let C be a nonempty approximatively compact, convex subset of locally convex Hausdroff topological vector space C, and suppose that $F \in \mathbf{V}_C{}^+(C, E)$ is a compact multifunction. Then for each continuous seminorm p on E there exists an $(x_0, y_0) \in F$ such that

$$p(x_0 - y_0) \leq p(x - y_0) \text{ for all } x \in C.$$

Proof: Consider the metric projection $Q_p : E \to 2^c$. Clearly, $Q_p(x)$ is nonempty, compact, and convex for every $x \in E$ and Q_p is an u.s.c. multifunction. Hence, $Q_p \in \mathbf{K}(E, C) \subset \mathbf{V}^+(E, C)$. Since it is clear that $\mathbf{V}^+$ is closed under composition, we have $Q_pF \in \mathbf{V}_C{}^+(C, C)$ and Q_pF is compact. Therefore, by Lemma 6.7, Q_pF has a fixed point. That is, there is a $y_0 \in Fx_0$ such that

$$x_0 \in Q_p y_0 = \{x \in C : p(y_0 - x) = d_p(y_0, C)\} \text{ (see [10, 11] for details).}$$

For a continuous multifunction $F : C \to 2^E$ Theorem 17 holds, where one cannot dispense with the lower semicontinuous. However, in Theorem 19, it is not needed that F be lower semicontinuous, only upper semicontinuity serves the purpose.

Example. Take $E = \mathbf{R}^2$ and $C = [0, 1] \times \{0\}$. Suppose $F \in \mathbf{V}(C, E)$ is given by

$$F(a,0) = F(a,0) \begin{cases} (0,1) \text{ if } a \neq 0. \\ [(0,1),(1,0), \text{ if } a = 0, \end{cases}$$

where $[A, B]$ stands for the closed line segment joining points A and B in the plane. Then F is not lower semicontinuous but $x_0 = (0,0)$ and $y_0 = (\frac{1}{2}, \frac{1}{2})$ satisfies the conclusion of Theorem 19.

Tecent work of Vetrivel, Veeramani and Bhattacharyya [20] is worth including. Here the multifunction F need not be convex-valued.

We need the following definition.

Definition 20: Let $F = F_1 \circ F_0$. The mapping $F : C \rightarrow E$ is said to have property (A) if and only if whenever there exists a $y \in F_0(x)$ for some $x \in C$ such that $d(y, C) = d(F_0 x, C)$, then $d(F_1 y, C) = d(F_0 x, C)$.

They proved the following :

Theorem 21: Let C be a nonempty approximatively compact convex subset of locally convex Hausdorff topological vector space E. Let $F : C \rightarrow E$ be a multifunction satisfying property (A), where F_1 and F_0 are closed convex-valued continuous multifunctions and $F_0(C)$ is relatively compact. Then there exists an $x \in C$ such that

$$d(x, fx) = d(Fx, C).$$

In their proof, they used the Lassonde's fixed point theorem [7], whereas others used the Himmelberg's theorem [6].

A multifunction $F : X \rightarrow Y$, is denoted by $R(X, Y)$, if there exists two topological spaces X and Y such that (i) $F : X \rightarrow Y$ and (ii) $F = F_2 \circ F_1$, where F_2 and $F_1 \in R$, [$F \in R$ means $F : X \rightarrow Y$, F is usc and Fx is nonempty compact convex subset of Y (Y is a convex set in a topological vector space)].

The following is due to Lassonde [7].

Theorem 22: Let C be a nonempty convex subset of a locally convex Hausdorff topological vector space *X*. Then any compact multifunction $F \in R(C, C)$ has a fixed point.

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 43-52

FARTHEST POINTS IN NORMED LINEAR SPACES

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(Received 19-10-95)

INTRODUCTION

We recall that if G is a bounded subset of a (real or complex) normed linear space $Ex \in E$ and $g_0 \in G$, then g_0 is called a farthest point to x in G whenever

$$(1) \qquad \| g_0 - x \| = \sup_{g \in G} \| g - x \|$$

In section 2 of the present paper we shall give some characterizations of farthest poings to x in G. In section 3 we shall give some applications of farthest points in $C(Q)$ and $C_R(Q)$ where Q is compact.

CHARACTERIZATIONS OF FARTHEST POINTS.

$$\text{Let } F_G(x) = \left\{ g_0 \in G : \| g_0 - x \| = \| g_0 - x \| = \sup_{g \in G} \| g - x \| \right\}$$

In this section we shall consider some conditions that are related to farthest points from a bounded subset G of E.

Theorem 1: Let G be a bounded subset of a normed linear space E, $x \in E$ and $g_0 \in G$. Then $g_0 \in F_G(x)$ if and only if there exists an $f_0 \in E^*$ such that

$$(2) \qquad f_0 \in \mathcal{E}(S_{E^*})$$

$$(3) \qquad f_0(g_0 - x) = \sup_{g \in G} \| g - x \|$$

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To prove this theorem we need the following results.

Lemma 1 [2] : Let M be an extremal subset of a closed convex set A in a topological linear space L. Then

(4) $\mathcal{E}(M) = \mathcal{E}(A) \cap M,$

Where $\mathcal{E}(M)$ denotes the set of all extremal points of M.

Lemma 2 [7]: Let E be a normed linear space and let F be a nonempty convex subset $\{ x \in E : \| x \| = 1 \}$. Then the set

(5) $M_F = \bigcap_{x \in F} \{ f \in E^* : \| f \| = 1, f(x) = 1 \}$ is a nonempty extremal subset of the cell $S_E^* = \{ f \in E^* : \| f \| \leq 1 \}$ endowed with $\partial(E^*, E)$.

Corollary 1 [7] : Let E be a normed linear space and $x \in E$, $x \ddagger 0$. Then the set

(6) $M_X = \{ f \in E^* : \| f \| = 1, f(x) = \| x \| \}$

is a nonempty extremal subset of S_E^* endowed with $\partial(E^*, E)$.

Proof of the Theorem (1): Let $g_0 \in F_G(x)$. Then, by the theorem 3.1 [3], there exists $f \in E^*$ such that $\| f \| = 1$.

$$f(g_0 - x) = \sup_{g \in G} \| g - x \|$$

By corollary 1, the set $M = \left\{ f \in E^* : \| f \| = 1, f(g_0 - x) = \sup_{g \in G} \| g - x \| \right\}$

is a nonempty extremal subset of S_E^* endowed with $\partial(E^*, E)$. So by Krein-Milman theorem, the set of $\mathcal{E}(M)$ is a nonempty and hense, by Lemma 1, there exists an $f \in E^*$ such that (2) and (3) hold.

Conversely, assume that (2) and (3) hold. Then, by (2) and (3),

$$\| g_0 - x \| \geq \| g - x \| \qquad (g \in G)$$

whence $g_0 \in F_G(x)$.

Corollary 2: Let G be a bounded subset of normed linear space E, $x \in E$ and $g_0 \in G$. Then the following statements are equivalent :

(a) $g_0 \in F_G(x)$

(b) There exists $f \in E^*$ such that f satisfies (2).

$$|f(g_0 - x)| = \sup_{g \in G} \|g - x\| \,, \tag{7}$$

$$|f(g_0 - x| \geq |f(g - x) \qquad (g \in G) \tag{8}$$

(c) There exists $f \in E^*$ such that f satisfies (2), (7) and

$$\mathrm{Re}\, f(g_0 - g)\overline{f(g_0 - x)} \geq 0 \tag{9}$$

Proof: Let $g_0 \in F_G(x)$. Then, by theorem 1, there exists $f \in E^*$ sych that (2) and (7) hold. by (7) and (2), we have

$$|f(g_0 - x)| \geq \|g - x\| \geq |f(g - x)| \qquad (g \in G)$$

Consequently, (a) $\Rightarrow$ (b). To prove (b) $\Rightarrow$ (c) assume that we have (b). Then by (8),

$$|f(g_0 - x)|^2 \geq |f(g - x)|^2 = |f(g - g_0)|^2 + f(g_0 - x)|^2$$

$$+ 2\, \mathrm{Re}\, f(g - g_0)\, \overline{f(g_0 - x)}$$

$$\geq |f(g_0 - x)^2 + 2\,\mathrm{Re}\, f(g - g_0)\,\overline{f(g_0 - x)}.$$

whence Re $f(g_0 - g)\,\overline{f(g_0 - x)} \geq 0$. (c) $\rightarrow$ (a) is trivial.

Example: Let $E = R^2$, the set

$G\ \{(x_1, x_2) : 0 \leq x_1 \leq 2; -1 \leq x_2 \leq 1\}$

and the point $x = (2.2)$. Then $(0, -1) \in f_G(x)$.

3. APPLICATIONS OF FARTHEST POINTS IN THE SPACES $C(Q)$ AND $C_R(Q)$

Theorem 3 : Let $E = C(Q)$ (Q compact), G a bounded subset of $x \in E$ and $g_0 \in G$ then $g_0 \in F_G(x)$ if and only if there exists a Radon measure μ (Real of Complex) on Q such that

(10) $|\mu|(Q) = 1$

(11) $$\frac{d\mu}{d|\mu|}(q) \in C(Q)$$

(12) $$g_0(q) - x(q) = \left[sign\, \frac{d\mu}{d|\mu|}(q)\right] \sup_{g \in G} \sup_{q \in Q} |g(q) - x(q)| \quad [q \in S(\mu)],$$

where (11) means that $\frac{d\mu}{d|\mu|}$ can be made continuous on Q by changing its values on a set of $|\mu|$ measure zero, in (12) $\frac{d\mu}{d|\mu|}$ is continuous function and $s(\mu)$ is the carrier of the measure μ.

Proof By theorem 3.1 [3] $g_0 \in F_G(x)$ if and only if there exists a Random measure μ on Q such that we have (10) and

(13) $$\int_Q [g_0(q) - x(q)]\, d\mu(q) = \sup_{g \in G} \sup_{q \in Q} |g(q) - x(q)|$$

By (10), (13) and $g_0 - x \neq 0$, we show that

(14) $$\frac{d\mu}{d|\mu|}(q) = \frac{\overline{g_0(q) - x(q)}}{\sup_{g \in G} \sup_{q \in Q} |g(q) - x(q)|} \quad |\mu| - a.e \text{ on } Q$$

For that assume that there exists a set $A \subset Q$ with $\mu(A) > 0$ such that

$$(15) \qquad \frac{d\mu}{d|\mu|}(q) \neq \frac{\overline{g_0(q)-x(q)}}{\sup_{g\in G}\sup_{q\in Q}|g(q)-x(q)|}\,|\mu|\text{-}a.e \text{ on } Q$$

then $Re\left[\frac{d\mu}{d|\mu|}(q)\{g_0(q)-x(q)\}\right] < \sup_{g\in G}\|g-x\|\ |\mu|$ - $a.e.$ on A.

otherwise , by taking into account that

$$(16) \qquad \left|\frac{d\mu}{d|\mu|}(q)\right| = 1 \quad |\mu| \text{ -}a.e \text{ on } Q,$$

there would exist a set $A_1 \subset A$ with $\mu(A_1) \supset > 0$ such that

$$\sup_{g\in G}\|g-x\| < Re\left[\frac{d\mu}{d|\mu|}(q)\{g_0(q)-x(q)\}\right]$$

$$\leq \left|\frac{d\mu}{d|\mu|}(q)\ \{g_0(q) - x(q)\}\right|$$

$$\leq \|g - x\|$$

$$\leq \sup_{g\in G}\|g-x\|\ |\mu| \text{ - } a.e. \text{ on } A_1.$$

which implies that $\frac{d\mu}{d|\mu|}(q)\ [g_0(q) - x(q)]$ is real and positive $|\mu|$ -$a.e$ on A_1 and hence

$$\frac{d\mu}{d|\mu|}(q)[g_0(q) - x(q)] = \sup_{g\in G}\|g-x\|\ |\mu| \text{ - } a.e \text{ on } A_1.$$

$$\frac{d\mu}{d|\mu|}(q) = \frac{\sup\limits_{g\in G} \| g - x \|}{g_0(q) - x(q)} = \frac{\overline{g_0(q) - x(q)}}{\sup\limits_{g\in G} \| g - x \|} | \mu | - a.e\text{on } A_1$$

which contradicts (15) Consequently,

$$\operatorname{Re} \int_Q [g_0(q) - x(q)] d\mu(q) = \operatorname{Re} \int_Q [g_0(q) - x(q)] \frac{d\mu}{d|\mu|}(q) d\mu(q)$$

$$= \int_Q \operatorname{Re} \frac{d\mu}{d|\mu|}(q) \{g_0(q) - x(q)\} d\mu(q)$$

$$< \int_Q \sup_{g\in G} ||g - x|| d|\mu|(q)$$

i.e. $\quad < \sup\limits_{g\in G} \| g - x \|$

which contradicts (13). Hence we obtain (14). By changing the values of $\frac{d\mu}{d|\mu|}$ on a set of $|\mu|$ measure zero such that (14) holds everywhere on Q, we obtain (11). By (16).

(17) $\quad |\frac{d\mu}{d|\mu|}| = 1 \qquad [q \in S(\mu)]$

Indeed, if there exists a $q_0 \in S(\mu)$ such that $|\frac{d\mu}{d|\mu|}(q) \neq 1$,

then, by taking an open neighbourhood U of q_0 such that

$|\frac{d\mu}{d|\mu|}(q)| \neq 1 \ (q \in U)$, we obtain, since $q_0 \in S(\mu)\ |\mu|\ (U) > 0$ which is a contradiction to (16). Thus, by (14) (everywhere on Q) we obtain (12).

Conversely, let the conditions (10) - (12) be satisfied. Then, by (12), (11) and (10),

$$\int_Q [g_0(q) - x(q)]d\mu(q) = \int_Q \sup_{g\in G} \max_{q\in Q} [g(q) - x(q)] \; sign \frac{d\mu}{d|\mu|}(q)d\mu(q)$$

$$= \sup_{g\in G} ||g - x|| \int_Q Sign \frac{d\mu}{d|\mu|}(q) \; \frac{d\mu}{d|\mu|}(q)d|\mu|(q)$$

$$= \sup_{g\in G} ||g - x|| \int_Q |\frac{d\mu}{d|\mu|}(q)| \; d|\mu|(q)$$

$$= \sup_{g\in G} ||g - x||,$$

which complets the proof.

For a compact space Q, we denote by $C_R(Q)$ the space of all continuous real valued function on Q, endowed with the ususal vector operations and with norm $||x|| = \max_{q\in Q} |x|(q)$ ·

Theorem 4 : Let $E = C_R(q)$ *(Q compact),* G a bounded subset of E, $x \in E$ and $g_0 \in G$. Then $g_0 \in F_G(x)$ if and only if there exist two disjoint sets $Y^+_{g_0}$ *and* $Y^-_{g_0}$ closed in Q and a Radon measure μ on Q such that

(18) $|\mu|(Q) = 1$,

μ is nondecreasing on $Y^+_{g_0}$ *noninvrea* $\sin g$ *on* $Y^-_{g_0}$ and

(19) $S(\mu) \subset Y^+_{g_0} \cup Y^-_{g_0}$ [$S(\mu)$ - the carrier of μ]

$$(20)\ g_0(q) - x(q) = \begin{cases} \sup\limits_{g\in G} \max\limits_{q\in Q} |g(q)-x(q)|, q \in Y^+_{g_0} \\ -\sup\limits_{g\in G} \max\limits_{q\in Q} |g(q)-x(q)|, q \in Y^-_{g_0} \end{cases}$$

Proof : By theorem 3.1 [3], $g \in F_G(x)$ if and only if there exists a Radon measure μ on Q such that (10) and (13) hold. We shall show that these conditions are equivalent to (18) - (20). Assume that we have (10) and (13). Now define

$$(21)\qquad Y^+_{g_0} = \left\{ q \in Q : g_0(q) - x(q) = \sup_{g\in G} \max_{q\in Q} |g(q)-x(q)| \right\}$$

$$(22)\qquad Y^- = \left\{ q \in Q : g_0(q) - x(q) = -\sup_{g\in G} \sup_{q\in Q} | g(q)-x(q) | \right\}$$

Then, by definition, $Y^+_{g_0}$ and $Y^-_{g_0}$ are disjoint closed sets in Q and so we obtain (20). To prove (19) Let μ be decreasing on $Y^+_{g_0}$. Then there would exist a set $A \subset Y^+_{g_0}$ with $\mu(A) > 0$ such that

$\mu(A) < |\mu|(A)$. So

$$\int_A |g_0(q) - x(q)| d\mu(q) = \int_A \sup_{g\in G} \max_{q\in Q} |g_0(q) - x(q)| d\mu(q)$$

$$= \sup_{g\in G} ||g-x|| \mu(A)$$

$$< |\mu|(A) \sup_{g\in G} ||g-x||$$

whence by (18)

$$\int_Q [g_0(q) - x(q)]d\mu(q) < \int_Q \sup_{g\in Q} ||g - x|| d|\mu|(q) = \sup_{g\in G} ||g - x||,$$

which contradicts (13). Hence μ is nondecreasing on $Y^+_{g_0}$. Similarly it can be shown that μ is nonincreasing on $Y^-_{g_0}$.

If there exists a $q_0 \in S(\mu)$ such that $Q \notin Y^+_{g_0} \cup Y^-_{g_0}$, then

$$|g_0(q_0) - x(q_0) < \sup_{g\in G} ||g - x||, .$$

by taking a neibourhood U of q_0 such that

$$|g_0(q)\text{-}x(q)\ | < \sup_{g\in G} ||g - x||, \qquad (g \in U),$$

we would have $|\mu| (q) > 0$, since $q_0 \in S(\mu)$. So

$$\int_U [g_0(q) - x(q)]d\mu(q) < \int_U |g_0(q) - x(q)|\ d|\mu|(q)$$

$$\leq \int_U \sup_{g\in G} ||g - x|| d|\mu|(q)$$

whence, by (10),

$$\int_Q [g_0(q) - x(q)]d\mu(q) < \int_Q \sup_{g\in G} ||g - x|| d|\mu|(q) = \sup_{g\in G} ||g - x||.$$

which contradicts (13). Hence $S(\mu) \subset Y^+_{g_0} \cup Y^-_{g_0}$.

Conversely, assume that there exists two disjoint closed sets $Y^+_{g_0}$ and $Y^-_{g_0}$ in Q and a randon measure μ on Q such that (18) - (20) hold.
Then by (19), (20) and (18).

$$\int_Q [g_0(q) - x(q)]d\mu(q) \le \int_{s(\mu)\cap Y^+_{g_0}} \sup_{g\in G} \max_{q\in Q} |g(q) - x(q)|\, d\mu(q)$$

$$+ \int_{s(\mu)\cap Y_{g_0}} -\sup_{g\in G} \max_{q\in Q} |g(q) - x(q)|\, d\mu(q)$$

$$\int_0 -\sup_{g\in G} \max_{q\in Q} |g(q) - x(q)|\, d\mu(q) = \sup_{g\in G} \|g - x\|$$

Thus gives (13). Thus (18) - (20) are none other then (10) - (13).

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Journal of Natural & Physical Sciences Vol. 9-10 (1995-96) pp 53-60

μ AND λ INVARIANTS OF A p-ADIC MEASURE

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(Received 14-04-95)

ABSTRACT

The Iwasawa Invariants of p-adic measure (a power series $F(x)$) will be characterized by Kummer-type congruences. As an application, a new generalization of the Kummer congruence is obtained. This is expressed in new compact notation.

Mathematics subject classifications (1991) 11B68, 11S40

Keywords : Kummer congruences, Iwasasa μ and λ Invariants, Bernoulli numbers, p-adic measure.

INTRODUCTION

Let p be a (fixed) prime number and put $q = 4$ for $p = 2$ and $q = p$ for $p > 2$. Let $\mathfrak{O}$ be the ring of intigers of a finite extention K of Q_p , with a local parameter π , $[k{:}Q_p] = e$. Denote by ord_π the π-adic valuation on K (i.e., $ord_\pi(x) = t$ means $x \in \pi^t\mathfrak{O}/\pi^{t+1}\mathfrak{O}$). Iwasawa invariants, $\mu(F)$ and $\lambda(F)$, of a power series $F(x) = \sum_{i=0}^{\infty} a_i x^i \in \mathfrak{O}[[x]]$ are defined by,

$$(1.1)\quad \mu(f) = min\{ord_\pi(a_i) : i \geq 0\} \text{ and } \lambda(f) = min\{i : \mathrm{ord}_\pi(a_i) = \mu(f)\}.$$

In his proof of the Ferrero - Washington theorem ($\mu = 0$), Sinnot [9] showed that the μ-invariants of p-adic measure and its Γ-transform are equal. In a related paper [7], Kida proved a theorem concerned with the λ-invariants of a p-adic measure supported on $1 + pZ_p$. Metsänkylä [8] gave a relation between Kummer type congruences and Iwasawa invariants provided that $\lambda < p$ and $\mu = 0$. In this paper Iwasawa invariants will be characterized by Kummer-type congruences and a new generalization of Kummer congruence is proven.

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2. The research upon which this paper is based was started at the University of Illinois at Urbana. I should like to express my appreciation to the staff of the math department there for their help and encouragement.

Define $G_c\,(j,F,n) \in K$ for $j,n \geq 0$ by

$$(1.2) \quad G_c\,(j,F,n) = \left(q_j^{-1}\Delta_c\right)F(\beta\alpha^n - 1),$$

where $\alpha \in qZ_p$ and $\beta \in 1 + \pi\mathfrak{O}$ are constants, $c > 0$ is an integer, $\left[q_J^{-1}\Delta_c\right]$ is the binomial coefficient operator acting on $n(\binom{x}{j} = \frac{x(x-1)\ldots(x-j+1)}{j!})$ and difference operator Δ_c is defined by $\Delta_c x_n = x_{n+c} - x_n$.

Remark : It is clear that $G_c\,(j,F,n)$ depends on λ, β and the infinite sum, $f(\beta\lambda^n - 1)$,converges in the π-adic topology because $\beta\alpha^n - 1 \in \pi O$.

(1.3) Theorem : (i) $G_c\,(j,F,n) \in \mathfrak{O}$ and $G_c\,(j,F,n)$ mod $\pi\mathfrak{O}$ is independent of n.

(ii) if $(c,p) = 1$ and $\alpha \notin 1 + pqZ_p$, then

$\mu(F)=min\{ord_\pi(G_c(j,F,n)) : j \geq 0\}$ and $\lambda(F)=min\{j : ord_\pi(G_c\,(j,F,n)=\mu(F)\}$.

PROOF OF THE THEROEM:

Let $s(l,\,m)$ and $S(r,\,h)$ be the Stirling numbers of first and second kinds respectively. The binomial coefficient operator, $\left[q_J^{-1}\Delta_c\right]$, is defined by

$$(2.1) \quad \left(q_J^{-1}\Delta_c\right)\tau^n = (1/j!)\left(\sum_{i-0}^{j} s(j,i)q^{-i}D_c^i t^n\right),$$

where $\Delta^i_c(\tau^n) = \sum_{i-0}^{j}\binom{i}{k}(-1)^{i-k_\tau n+kc} = (\tau^c-1)^i\tau^n$ Hence τ^n is an eigenvector of $\left(q_j^{-1}D_c\right)$ with eigenvalue $\binom{\eta(\tau)}{j}$ where $\eta(\tau) = (\tau^c - 1)/q$

Define $A(t,k,n) \in K$ by

(2.2) $$A(t,k,n) = \left(q_t^{-1}D_c\right)(\beta\alpha^n - 1)^k \quad \text{for } t,\ k,\ n \geq 0.$$

For brevity we write *mod* p^r (*mod* π) for *mod* p^rZ_p (*mod* $\pi\mathfrak{O}$)

(2.3) Lemma: (*i*) $ord_{\pi}A(t,k,n) \geq max\ \{0,\ k\text{-}2et\text{-}ord_{\pi})\ t!)\}$.

(ii) $A(t,k,n)$ *mod* π is independent of n.

(*We write* $A(t,k) \not\equiv A(t,k,n)$ *mod* π.)

(iii) $A(t,k) \not\equiv 0\ (mod\ \pi)$ if $k > t$.

(iv) $A(k,k) \not\equiv r^k c^k\ (mod\ \pi)$, where $\alpha = 1 = \alpha q$.

In particular (a) $A\ (t,k,n) \to 0$ as $k \to \infty$ and

(b) $A(k,k) \not\equiv 0 (mod\ \pi)$ if $\alpha \notin 1 + pqZ_p$ and $(c,p) = 1$.

Proof: (i) From the definition $ord_{\pi}A(t,k,n) \geq k\text{-}2et\text{-}ord_{\pi}).t\ !)\}$.

Since $\alpha \in 1 + pZ_p$, $\eta(\alpha^i) \notin Z_p$ and hence $\binom{\eta(\alpha^i)}{t} \in Z_p$. We get from $(\beta\alpha^n\text{-}1)^k = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(\beta\alpha^n)^i$ that

$$A(t,k,n) \overset{\cdot}{=} \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(\beta\alpha^n)^i\binom{\eta(\alpha^i)}{t} \in \mathfrak{O}\ \square.$$

(ii) We have $A(t,k,n) \not\equiv \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\binom{n(\alpha^{i})}{t}(\bmod \pi)$ (because $(\beta\alpha^{n} \in 1 + \pi\,\mathfrak{O})$ from which it follows that $A(t,k,n) \bmod \pi$ is independent of n□.

(iii) and (iv) Recall that the p-adic exponential function, $exp_p(x)$, (we write e^x for simplicity) converges for all $x \in qZ_p$. Hence for all $x \in Z_p$,

$$(e^{qX} - 1)^{h} = (qx)^{h}\,(1 + \sum_{r=1}^{\infty} qb_r x^{r}) \text{ with } b_r \,\varepsilon\, Z_p .$$

(Because $e^{qX} - 1 = qx(1 + \sum_{r=1}^{\infty} \frac{(qx)^{r}}{(r+1)!})$, $ord_2\,(4^{r}/(r+1)!) \geq 2r - r > 0$ and if $p \neq 2$, $ord_p\,(p^{r}/(r+1)!) \geq r - (r/(p-1)) > 0$.) Since e^{q} is a topological generator for $1 + qZ_p$ there is a $d \in Z_p$ such that $\alpha = e^{qd}$ and $d \not\equiv \gamma\,(mod\,p)$. Hence

$$\eta(\alpha^{i})^{h} = (e^{qdci} - 1)^{h}/q^{h} = (dci)^{h}\,(1 + \sum_{r=1}^{\infty} qb_r\,(dci)^{r}).$$

$$\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\,\eta(\alpha^{i})^{h} = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\,(dci)^{h}\,(1 + \sum_{r=1}^{\infty} qb_r\,(dci)^{r})$$

$$= k!\,d^{h}c^{h}\,(S(h,k) + \sum_{r=1}^{\infty} S(r+h,\,k)\,qb_r\,d^{r}c^{r}).$$

(Because $\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\,i^{m} = k!\,S(m,\,k)$.) Interchanging summation we get,

$$(2.4)\; A(t,\,k) \not\equiv \sum_{k=0}^{t}\frac{k!}{t!}\,s(t,h)\,d^{h}c^{h}\;S(h,\,k) + \sum_{r=1}^{\infty} S(r+h,\,k)\,qb_r\,d^{r}c^{r})(mod\,\pi).$$

Hence $A(t,\,k) \not\equiv \sum_{k=0}^{t}\frac{k!}{t!}\,s(t,h)\,d^{h}c^{h}\;S(h,\,k) \not\equiv \delta_{t,k}\,\gamma^{k}\,c^{k}\,(mod\,\pi)$ for all $k \geq t$.

(Because $s(m, j) = 0 = S(m, j)$ whenever $0 \le j < m$ and $d \not\equiv \gamma \,(mod\, p)$ ■

Remark: Suppose $\gamma \not\equiv 1 \ (mod\, p)$ and $c \not\equiv 1 \ (mod\, p)$. if $k < t < p$ then $A(t,k) \not\equiv 0 \ (mod\ \pi)$.

Proof of the theorem: From definitions 1.2 and 2.2, one gets $G_c(j,F,n) = \left(q_j^{-1}\Delta_c\right) \sum_{i=0}^{\infty} a_i(\beta\alpha^n - 1)^i = \sum_{i=0}^{\infty} a_i A(j, i, n)$ and then apply 2.3 to get,

$$(2.5) \quad G_c(j,F,n) = \sum_{i=0}^{j} a_i A(j, i, n) \ (mod\ \pi).$$

Then 1.3(i) follows from (i) and (ii) of 2.3.

We can assume that $\mu(F)=0$ (Because, if $\mu(F) > 0$, replace $F(x)$ by $E(x) = \pi^{-\mu(F)} F(x)$, then $\mu(E) = 0$ and $\pi^{\mu(F)} G_c(j, E, n) = \pi^{\mu(F)} \left(q_j^{-1}\Delta_c\right) E(\beta\alpha^n - 1) = \left(q_j^{-1}\Delta_c\right) F(\beta\alpha^n - 1) = G_c(j, E, n)$). But $\mu(F) = 0$ means $a_i = 0 \ (mod\ \pi)$ for all $i < \lambda(F)$ and $a_{\lambda(F)} \not\equiv 0 \ (mod\ \pi)$. Hence, one gets by using 2.5 and 2.3 that $G_c(j, F, n) \not\equiv 0 \ (mod\ \pi)$ for all $j < \lambda(F)$ and $G_c(\lambda(F), F, n) \not\equiv a_{\lambda(F)} A(\lambda(F), \lambda(F), n) \not\equiv 0 \ (mod\ \pi)$, from which it follows that $min\{ord_\pi(G_c(j,F,n)) : j \ge 0\} = 0 = \mu(F)$ and $\lambda(F)=min\{j : ord_\pi(G_c(j,F,n)=0\}$ ■

CONSEQUENCES AND APPLICATIONS

Let χ be a non-trivial even Dirichlet character with conductor rp^t, $t \ge 0$, $(r,p) = 1$ and $K = Q_p(\chi)$ .If $r = 1$ assume that $\chi(m) \ne \chi(m+p^k)$ for all $k \ge 0$. From Iwasawa theory [4-5] there is a power series $F(x, \chi) \in \mathfrak{O}[[x]]$ associated to χ such that $F(\zeta(1+rp)^s,\chi) = L_p(s,\chi)$, where $L_p(s,\chi)$ is a p-adic L-function associated to χ and $\zeta = \chi(1 + rp)^{-1}$. Set $\alpha = (1 + rp)^{-1}$ and $\beta = \zeta(1 + rp)$ and write $G(j, \chi, \eta)$ for $G(j, F, n)$. Then for all $n \ge 1$,

$$G_c(j, \chi, n) = \left(q_j^{-1}\Delta_c\right) F(\zeta(1+rp)^{1-n}, \chi, = \left(q_j^{-1}\Delta_c\right) L_p(1\text{-}n, \chi) = \left(q_j^{-1}\Delta_c\right) \beta_{\chi,n},$$

where $\beta_{\chi,n} = L_p(1 - n, \chi)$ (because $L_p(1 - n, \chi) = -\frac{1}{n}(1 - \chi\omega^{-n}(p)\,p^{n-1}) \times B_{,n,}\chi\omega^{-n}$, it represents an *extended generalized Bernoulli number*, where ω is the Teichmüller character). Let $K_c(j, \chi, n) = q^{-j}\nabla^j \beta_{n,\chi}$ then $K_c(j, \chi, n) = \sum_{i=0}^{j} i!\, G_c(i, \chi, n)\, S(j, i) \in \mathfrak{O}$. Hence we get Metsänkylä's Theorems 1 and 2 [8] as a consequence of 1.3.

(3.1) Theorem 1 & 2 [8]: Suppose that $F(x, \chi) = \sum_{i=0}^{j} a_i x^i$ and $0 \le h < p$. Then following three conditions are equivalent.

(i) $a_i \not\equiv 0 \ (mod\ \pi)$ for all $0 \le i \le h$.

(ii) $K_1(i,\chi,n) \not\equiv 0 \ (mod\ \pi)$ for all $0 \le i \le h$.

(iii) $K_{p-1}(i,\chi,n) \not\equiv 0 \ (mod\ \pi)$ for all $0 \le i \le h$.

Note : The fact that theorem 1.3 is independent of c gives the equivalence of (ii) and (iii)

Since $K_c(j,\chi,n) \not\equiv \sum_{i=0}^{p-1} i!\, G_c(i, \chi, n)\, S(j, i) \ (mod\ \pi)$ and $S(p\text{-}1+j, i) \not\equiv S(j, i) \ (mod\ p)$ whenever $j > 0$, we get the following result.

(3.2) Theorem (Periodicity of Kummer Congruences)

If $j, j' > 0$ and $j \not\equiv j' \ (mod\ p\text{-}1)$ and n, n' arbitrary then

$$K_c(j, \chi, n) \not\equiv K_c(j, \chi, n) \ (mod\ \pi).$$

Since $\mu(\omega^m) = 0$ [1] one gets by applying Theorem 1.3 with $\chi = \omega^m \neq 1$.

(3.3) Theorem (Generalized Kummer Congruence):

(i) $G_c(j, \omega^m, n)\ \varepsilon\ Z_p$ and $G_c(j, \omega^m, n)$ *mod p* is independent of *n*.

(ii) If $p \nmid c$ and $2 \mid m$ then there exists $i \geq 0$ such that $G_c(i, \omega^m, n) \not\equiv 0$ $(mod\ p)$ and $\lambda(\omega^m) = min\ \{ j : G_c(j, \omega^m, n) \not\equiv 0\ (mod\ p)\}$.

One can deduce the classical Kummer congruence from $K_c(j, \chi, n) \in \mathfrak{D}$ as follows : Set $c = p-1$ $(p > 2)$ and $\chi = \omega^n\ (\neq 1)$ then $p^j K_{p-1}(j, \omega^n, n) \in p^j Z_p$.

$$\text{i.e.} \qquad \Delta^j \beta_{n,\omega^n} = -\sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} \frac{(1-p^{n-1})}{n+k(p-1)} B_n \not\equiv 0\ (mod\ p^j Z_p).$$

Theorem 1.3 together with a result in [3] can be used to prove the Iwasawa criterion [6] for $\mu(\omega^m) = 0$. Lemma 2.3 is also important its own right. For example, one can use 2.3 to give a proof of Kida's theorem [7] on λ-invariants of Γ-transform of a p-adic measure [3].

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 61-71

AEROMYCOFLORA OVER POTATO FIELDS

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(Received 03-04-95 and after revision 10-8-96)

ABSTRACT

Aeromycology over potato fields at Kurukshetra was studied consecutively for two years in relation to environmental conditions using culture plate method. A total of 25 and 29 fungal forms were trapped from the air in 1987-88 and 1988-89 crop season. Cladosporium cladasporioides was the chief constituent contributing 59.33% and 57.93% of the total fungi in the two respective years. The other dominant fungi in the two successive years were those of Alternaria, alternata, Aspergillus, flavus, A. niger, Cladosporium, herbarum, Epicoccum, nigrum, Fusarium equisetum, Penicillium, citrinum, P., cyclopium and Trichothecium, roseum. In the diurnal cycle the fungi showed an evening tendency. Rain has profound effect on the aeromycoflora.

Key words : Aeromycoflora, Cladosporium cladosporioides, Potato.

INTRODUCTION

Aerobiological studies conducted in various parts of India revealed the richness of airspora [15]. Standing vegetation has a great influence

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on airspora of any place and it changes with changes in weather [2]. In the present study aeromycoflora of potato fields was studied consecutively for two years in relation to environmental conditions.

MATERIALS AND METHODS :

Aeromycology of potato fields was recorded during 1987-88 and 1988-89 crop season at Kurukshetra (Haryana) by culture plate method. Four petri dishes containing Martin's agar medium were exposed for five minutes at one and half feet height in the potato fields at 6.00 a.m., 12.00 noon, 6.00 p.m. and 12.00 night. These plates were incubated at 25^{0}C and the number of fungal colonies appearing were counted and identified. Relative humidity and temperature were recorded by placing hygrographs and thermographs in the fields and rainfall was recorded from soil and water testing laboratory, Kurukshetra.

RESULTS:

1. **Components** : The spore content of air over the potato fields was rich both qualitatively and quantitatively. A list of fungal spore types and the percentage contribution to total aeromycoflora was given in Table 1. A total of 25 and 29 fungal forms trapped from the air in 1987-88 and 1988-89 crop season. Acremonium, implicatum, Alternaria, longipes, Bipolaris, hawaiiensis, Emericella nidulans and Epicoccum, purpurascens were restricted to first year. While Aspergillus, fumigatus, Fusarium, moniliforme, Myrothecium verrucaria, Nigrospora sphaerica, Penicillium purpurogenum and Periconia, saraswatipurensis were trapped only from the aeromycoflora of second year crop season. Of all the components Cladosporium cladosporioides was the chief constituent contributing 59.33% and 57.93% of the total fungi in the two respective years. The other abundant colonies in the two successive years were those of A. alternata, A. tenuissima, A. flavus, A., niger, C., herbarum, E.

nigrum, F., equiseti, P., citrinum, P., cyclopium, and T., roseum (Table 4 and 5)

2. **Diurnal and seasonal periodicities :** The fungal content of air over potato fields varied between 7.4 and 157.4 propagules per 100 cm^3 (Table 2) in the first year and between 6.5 and 117.8 propagules per 100 cm^3 in the second year (Table 3). In the diurnal cycle the fungi showed an evening tendency. On the basis of statistical analysis aeromycoflora varied significantly seasonally on one hand and diurnally it varied insignificantly on the other hand in 1987-88. While in 1988-89, aeromycoflora over potato varied insignificantly both seasonally and diurnally.

3. **Effect of weather :** Climatic changes profoundly influenced the aeromycoflora over potato fields. It appears from the data (Fig. 1) that high temperature and low relative humidity in the month of October and November favoured the aeromycoflora but comparatively low temperature and high humidity have adversely affected the aeromycoflora. Of all the climatic factors, rain is the single important factor that causes major fluctuations in the spore content of the air. The sampling taken after rains showed a decrease in the spore content in the air. In 1987-88 the first spell of rain i.e. 6.5mm was recorded on 14.12.87. It was found that it increased the spore load in the air which is evident from the third sampling taken on 25.12.87. The rain was frequent in the months of December 1988 and January 1989 with a prolonged spell of fog in the nights and mornings. This type of atmosphere decreased the spore load in the atmosphere and lowest counts were trapped in this period.

DISCUSSION:

In the diurnal cycle the aeromycoflora showed evening maxima and night and morning minima and this trend is in confirmity with earlier reports from India [1, 7, 12]. The fluctuations in the aeromycoflora can be attributed to prevailing environmental conditions. The maximum aeromycoflora on 25.12.87 indicates the effect of first rain in bringing more spores into the air was also observed by Sreeramulu and Ramalingam [14] and Malliah and Rao [8]. The least count on 21.01.88 can be attributed to the low temperature, foggy night and prolonged periods of dew on the surface of plants and other inanimate objects. In the next year the highest aeromycoflora was recorded on 30.01.88. This is due to absence of rain with optimum temperature and humidity in the atmosphere. The maximum aerospora in the dry weather was reported by Gregory [2] , Kamal and Singh [4,5] and Kamal and Verma [6]. The lowest aeromycoflora on 25.12.88 may be due to the washing down of atmospheric fungal spores along with dust particles as a result of 12 mm and 5.2 mm of rains on 22.12.88 and 24.12.88 respectively. The immediate decrease in spores after rains was observed by Hirst [3], Meredith [10] and Malliah and Rao [8]. Kumar and Gupta [7], while studying the aerospora over a potato field during September found lowest population of microorganisms. In subsequent seasons with gradual fall in temperature and rise in relative humidity the total population of airspora enhanced continuously. Our observations differ from that of Kumar and Gupta's because rainfall was absent during their period of study. While rainfall played an important role in quantifying the aeromycoflora.

Cladosporium is an ubiquitous fungus and its dominance in the fungal aerospora has been reported from different parts of our country [8, 9, 11]. The presence of Alternaria spores in abundance was obtained by Sreeramulu and Ramalingam [13] in the air over paddy fields. Aspergillus species were found to be present in the air mycoflora through out the period of investigation. Most of the Aspergillus colonies were trapped in the month of November and the number decreased in December and January. This is in concurrence with the report by Kamal and Singh [4].

Table 1 : Percentage contributions of different fungi to total aeromycoflora.

Sr. No.	Name of Organism	1987-88	1988-89
1.	*Acremonium implicatum*	1.77	-
2.	*Alternaria alternata*	10.55	5.68
3.	*A. longipes*	0.40	-
4.	*A. tenuissima*	2.10	0.62
5.	*Aspergillus candidus*	0.66	1.38
6.	*A. flavus*	3.15	4.32
7.	*A. fumigatus*	-	0.06
8.	*A. nidulans*	-	0.42
9.	*A. niger*	2.28	0.86
10.	*A. terreus*	1.02	0.36
11.	*A. versicolor*	0.21	0.42
12.	*Aspergillus sp*	-	0.24
13.	*Bipolaris hawaiiensis*	0.10	-
14.	*B. indica*	0.80	0.13
15.	*Cladosporium cladosporioides*	59.33	57.93
16.	*C. herbarum*	2.10	1.48
17.	*Curvularia lunata*	1.61	2.71
18.	*Emericella nidulans*	0.35	-
19.	*Epicoccum nigrum*	3.30	2.45
20.	*E. purpurascens*	0.35	-
21.	*Fusarium equiseti*	0.94	4.13
22.	*F. moniliforme*	-	0.96
23.	*F. pallidoroseum*	0.21	0.46
24.	*Myrothecium verrucaria*	-	0.30
25.	*Nigrospora sphaerica*	-	0.48
26.	*Penicillium citrinum*	2.36	2.39
27.	*P. cyclopium*	3.05	5.00
28.	*P. oxalicum*	0.03	0.90
29.	*P. purpurogenum*	-	0.36
30.	*Periconia saraswatipurensis*	-	0.24
31.	*Rhizopus oryzae*	0.05	1.42
32.	*Trichothecium roseum*	1.94	1.30
33.	White sterile form	1.19	2.06
34.	Black sterile form		0.13

Table 2 : Diurnal variations in the density of total aeromycoflora per 100 cm³ over potato fields in 1987-88 crop season.

Sampling time	**Sampling numbers and date**			
	I **5.11.87**	**II** **30.11.87**	**III** **26.12.87**	**IV** **21.1.88**
06.00 A.M.	29.4	18.5	12.0	7.4
12.00 A.M.	53.4	37.8	84.3	42.5
06.00 P.M.	119.4	120.6	157.4	51.4
12.00 P.M.	39.5	21.3	15.9	44.3

Two-way analysis of variance table

Source of variation	**DF**	**Sum of Squares**	**Mean Square**	**F**
Among means of treatments A	3	2196.662	732.220	1.132
Among means of treatments B	3	21329.182	7109.727	10.995*
Residual	9	5819.595	646.621	

A = Diurnal cycle; B = Time interval; DF = Degree of freedom. * Significant at 0.05 level.

Table 3 : Diurnal variations in the density of total aeromycoflora per 100 cm³ over potato fields in 1988-89 crop season.

Table 3 : Diurnal variations in the density of total aeromycoflora per 100 cm^3 over potato fields in 1988-89 crop season.

Sampling time	**Sampling numbers and date**			
	I **5.11.88**	**II** **30.11.88**	**III** **25.12.88**	**IV** **21.1.89**
06.00 A.M.	17.1	12.0	7.1	7.3
12.00 A.M.	19.4	32.8	11.3	8.8
06.00 P.M.	59.6	117.8	9.5	22.8
12.00 P.M.	23.3	11.6	6.5	11.1

Two-way analysis of variance table

Source of variation	**DF**	**Sum of Squares**	**Mean Square**	**F**
Among means of treatments A	3	3141.09	1047.03	2.105
Among means of treatments B	3	4532.221	1510.740	3.037
Residual	9	4475.879	497.319	

A = Diurnal cycle; B = Time interval; DF = Degree of freedom. * Significant at 0.05 level.

Table 4 : Percentage abundance of aeromycoflora over potato fields in the year 1987-88

S.n.	Name of organism	Sampling Numbers, Date and Sampling time															
		I				I				I7				I			
		5.11.88				30.11.88				25.12.88				21.1.89			
		6AM	12AM	6PM	12PM	6AM	12AM	6PM	12PM	6AM	12AM	6PM	12PM	6AM	12AM	6PM	12PM
1.	Acremonium implicatum	-	5.17	2.32	6.97	-	-	1.52	4.34	-	-	-	-	-	5.43	3.51	-
2.	Alternaria alternata	5.46	8.62	7.75	11.62	14.28	9.75	4.58	17.39	11.53	17.39	9.41	29.41	8.33	16.30	11.60	12.5
3.	A. longipes	2.34	3.44	-	2.32	-	-	-	-	-	-	-	-	-	-	-	-
4.	A tenuissima	-	-	2.32	6.97	-	2.43	1.52	-	-	2.17	1.76	-	-	4.34	3.57	3.12
5.	Aspergillus candidus	-	-	0.77	-	-	-	-	-	-	-	-	11.76	8.32	2.17	2.67	-
6.	A. flavus	8.59	5.17	3.10	4.65	14.28	7.31	1.52	13.04	-	-	1.76	11.76	-	-	0.89	2.08
7.	A. niger	3.90	3.44	1.55	6.97	4.76	2.43	2.29	8.69	7.69	1.08	1.17	5.88	-	-	0.89	1.04
8.	A. terreus	1.56	1.72	-	-	4.76	4.87	0.76	-	3.84	2.17	-	-	-	1.08	-	2.08
9.	A. versicolor	-	-	0.77	-	-	-	0.76	-	-	-	-	-	-	-	-	-
10.	Bipolaris hawaiiensis	-	1.72	-	-	-	-	-	-	-	-	-	-	-	-	-	-
11.	B. indica	-	1.72	-	-	-	2.43	3.05	-	-	1.08	-	-	-	-	0.89	-
12.	Cladosporium cladosporioides	46.87	53.44	69.76	37.20	42.85	56.09	67.93	39.13	53.84	59.78	70.58	35.29	20.83	56.52	48.21	54.16
13.	C. herbarum	-	-	-	-	-	-	5.34	-	-	-	4.70	-	-	-	4.46	4.16
14.	Curvularia lunata	-	-	-	-	-	-	1.52	-	-	4.34	2.35	-	-	2.17	-	5.20
15.	Emericella nidulans	-	-	-	-	-	-	-	-	3.84	-	-	-	4.16	-	3.57	2.08
16.	Epicoccum nigrum	-	-	2.32	2.32	-	-	2.29	4.34	7.69	1.08	2.35	-	33.33	9.78	8.92	9.37
17.	E. purpurascens	-	-	-	-	-	-	-	-	-	3.26	-	-	4.16	-	-	-
18.	Fusarium equiseti	8.59	-	2.32	4.65	4.76	-	-	-	-	-	-	-	-	-	-	-
19.	F. pallidoroseum	3.12	-	0.77	-	-	-	-	-	-	-	-	-	-	-	-	-
20.	Penicillium citrinum	8.59	6.89	3.87	9.30	4.76	2.43	-	8.69	3.64	-	-	-	8.32	-	-	2.08
21.	P. cyclopium	3.12	3.44	1.55	-	9.52	7.31	3.05	-	7.69	4.34	4.11	-	-	1.08	2.67	-
22.	P. oxalicum	-	-	-	-	-	-	-	-	-	-	-	-	4.16	-	-	-
23.	Rhizopus oryzae	1.56	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
24.	Trichothecium roseum	3.90	3.44	-	2.32	-	4.87	3.05	-	-	3.26	1.17	-	4.16	-	4.46	-
25.	White sterile form	2.34	1.72	0.77	4.65	-	-	0.76	4.34	-	-	0.58	5.88	4.16	1.08	0.89	2.08

Table 5 : Percentage abundance of aeromycoflora over potato fields in the year 1987-88

S.n.	Name of organism	Sampling Numbers, Date and Sampling time															
		I				I				17				I			
		5.11.87				30.11.87				25.12.87				21.1.88			
		6AM	12AM	6PM	12PM	6AM	12AM	6PM	12PM	6AM	12AM	6PM	12PM	6AM	12AM	6PM	12PM
1.	Alternaria alternata	1.35	3.57	6.20	2.24	-	9.85	7.36	4.00	-	4.08	2.43	3.57	3.12	3.47	4.04	12.50
2.	A tenuissima	-	-	1.28	-	-	0.70	0.73	-	-	-	-	-	-	-	1.01	2.08
3.	Aspergillus candidus	1.35	-	-	-	-	2.11	-	-	6.45	-	4.87	7.14	12.5	10.52	4.04	2.08
4.	A. flavus	-	1.19	1.02	3.37	30.76	7.74	1.28	10.00	25.80	2.04	-	21.42	18.75	-	-	10.41
5.	A. fumigatus	-	-	-	-	-	0.70	-	-	-	-	-	-	-	-	-	-
6.	A. nidulans	-	-	-	-	-	1.40	0.36	6.00	-	-	-	-	-	-	-	-
7.	A. niger	-	-	0.51	8.98	1.92	-	0.55	-	3.22	-	-	-	-	-	-	-
8.	A. terreus	-	-	-	-	-	-	0.92	-	-	-	-	-	3.12	-	-	-
9.	A. versicolor	5.40	-	-	-	3.37	-	-	-	-	-	-	-	-	-	-	-
10.	Aspergillus sp.	-	-	-	-	1.92	-	-	-	-	-	-	-	3.12	-	-	-
11.	Bipolaris indica	-	-	-	-	-	-	-	-	-	-	-	-	-	3.47	1.01	4.16
12	Cladosporium cladosporioides	39.18	64.28	65.11	56.17	40.38	59.15	72.00	32.00	19.35	44.89	63.41	53.57	31.25	28.00	37.37	43.75
13.	C. herbarum	5.40	-	-	-	-	-	0.92	-	-	-	2.43	-	-	17.47	6.06	4.16
14.	Curvularia lunata	5.40	3.57	1.02	-	-	6.33	3.68	6.00	-	4.08	-	-	-	3.47	-	-
15.	Epicoccum nigrum	-	-	0.51	-	-	1.40	0.73	-	16.12	20.40	-	-	-	3.47	13.13	8.33
16.	Fusarium equiseti	8.10	7.14	8.77	6.74	1.92	4.22	2.76	-	-	-	-	-	-	-	6.06	-
17.	F. moniliforme	-	1.19	1.53	-	-	2.11	1.47	-	-	-	-	-	-	-	-	-
18.	F. pallidoroseum	2.70	1.19	-	-	-	-	0.36	-	-	-	-	-	-	6.94	-	-
19.	Myrothecium verrucaria	-	-	-	-	-	-	-	-	-	-	2.43	-	-	-	4.04	-
20.	Nigrospora sphaerica	-	2.38	1.55	-	-	1.40	-	-	-	-	-	-	-	-	-	-
21.	Penicillium citrinum	13.5	8.33	4.12	8.98	-	-	0.18	4.0	-	-	-	-	3.12	-	-	-
22	P. cyclopium	-	3.57	-	4.49	7.69	2.11	2.20	12.00	16.12	14.28	9.75	7.14	12.5	21.00	20.20	2.08
23.	P. oxalicum	10.81	-	-	2.24	5.76	-	-	-	-	-	4.87	-	-	-	-	-
24.	P. purpurogenum	-	-	-	-	-	-	0.18	6.00	-	-	-	-	6.25	-	-	-
25	P. saraswatipurensis	-	-	-	-	-	-	-	-	-	8.16	-	-	-	-	-	-
26	Rhizopus oryzae	-	-	1.02	-	7.69	8.70	1.47	2.00	6.45	2.04	7.13	-	3.12	-	-	-
27.	Trichothecium roseum	-	1.19	1.02	4.49	-	-	1.28	2.00	-	-	-	-	3.12	-	3.03	4.16
28.	White sterile form	6.75	2.38	3.61	1.12	-	-	0.73	16.00	6.45	-	-	7.14	-	-	-	2.08
29.	Black sterile form	-	-	0.51	-	-	-	-	-	-	-	4.87	-	-	-	-	-

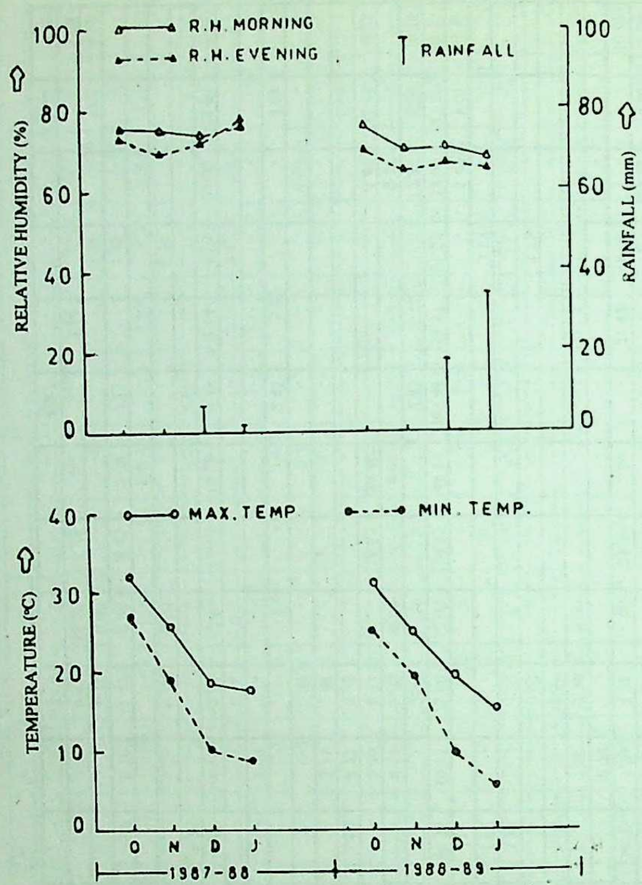

Fig.1 : Meteorological data of Kurukshetra during October, 1987 to January, 1988 and October, 1988 to January, 1989

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 73-76

A NOTE ON RAINGUSH PHENOMENON

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(Received 15-09-96 and after rivision 15-01-97)

Raingush is one of the important meteorological phenomenon. It is not very commonly observed in plains but relatively more frequent in hilly regions. During raingush we have abnormally high rain intensity many times more than that normally observed during heavy rain fall. It is more oftenly observed just after the lightning [1]. Though the scientific observations are not available on meteorological parameters proceeding and preceding raingush, Levin and Ziv [2] have related electric field variation at the time of lightning with occurrence of raingush. Thus it may be worthwhile to look for electrical processes of thunderclouds before, during, and after the lightning discharge.

Many electrical phenomenon those occur in the atmosphere can be explained on the basis of charging mechanism, e.g. airplanes speeding through the atmosphere are subjected to ventilation by ions and will charge to equilibrium potential of several thousand volts [6]. The earth surface is constantly ventilated by winds containing natural ions and is charged accordingly. This is perhaps thought to be one of the possible sources of atmospheric electricity [5]. Cloud drops become charged when they fall through the ionized atmosphere or are ventilated by strong updrafts winds that accompany the growth of convective clouds. Levin and Ziv [2] used a time dependent model coupling the growth of the particles in a cloud with the electrical development. To explain the phenomenon of raingush, they have assumed that the polarization charging is the only mechanism for cloud electrification. According to this mechanism , all the hydrometeors are polarized by downward fair weather electric field. They have negative charge in lower portions and the positive charge in the upper ones because of the gravitational separation. (Fig. 1) . The bigger drops reach the lower parts of the cloud with a net negative charge while the smaller droplets with net positive charge remain in upper region of

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the cloud. Thus a net positive charge is concentrated at the top of the cloud and a net negative charge become concentrated at the bottom. These concentrations cause an increase in the electric field which results in larger charge separation. This process continues until the electric field becomes strong enough to oppose the gravitational forces which help in the floatation of the particles or decrease their fall velocities.

In an electrified cloud a sudden growth of precipitation is often observed in association with lightning. Apparently, there seems to be a close association between the occurrence of lightning, a rapid growth of precipitation particles in the cloud (as may be concluded from the intensification of radar echo) and the sudden onset of a heavy precipitation at the ground beneath the cloud few minutes after the lightning [3,4]. This provides a basis for the hypothesis that the coalescence of cloud droplets into precipitating particles in the form of rain or hail, may be largely enhanced by the electric field and high charge densities in the vicinity of lightning channel. In the above context, Levin and Ziv [2] shown that the precipitating particles of the size of 200 μ m when within the cloud particles of the constant size of 10 μ m grow to 300 μ m in a time of about 300 sec. Thus the fall velocities also increase. Simultaneously, there is an increase in the charge on the particles and electrical field as shown by them {see fig 2(a), (b) reproduced herewith the permission}. Both the growth of charge on the particles and the electric field decrease the terminal velocity of large particles which are negatively charged. This also reduces the collision rate and rate of collection of smaller particles. This decreased number of collisions results in a reduced rate of charge accumulation on the large particles as well as a reduction in the rate of growth of the electric field. During the growth of the electric field, the precipitation rate increases as the result of an increase in both the radii of their drops and their fall velocities [Fig. 2(a)].

When the lightning occurs, a large fraction of charge is neutralized and electric field is decreased considerably. Thus the electrical forces those are responsible to hold the precipitation particles together are weakend drastically i.e. the electric field is not large enough to suspend them against the gravitational force, hence the droplets fall down suddenly in the form of rain, resulting a dramatic increase in the precipitation rate. Thus the raingush is observed. The increase in the precipitation rate is followed by further interaction among the particles, more charge separation, and a subsequent fast recovery of the field. The increase in the field strength again slows down the fall velocities of the particles and reduces precipitation rate. This reduction continues until the field again reaches the lightning threshold and results in second raingush.

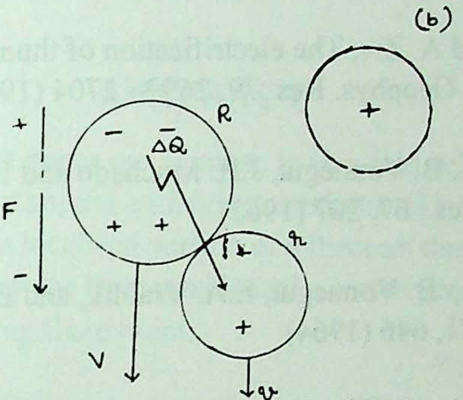

Fig. 1 Sequence of events during the polarization charging of cloud hydrometeors. (a) During contact (b) after separation.

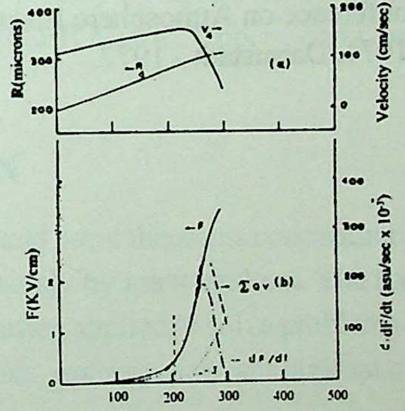

Fig. 2 Changes with time (a) The radius of precipitation particles R and their fall velocities *v* (b) The electric field F and rate of change of electric field $\frac{dF}{dt}$

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 77-86

FIXED POINT THEOREMS FOR MULTIFUNCTIONS

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(Received 10-10-96)

ABSTRACT

In the present paper our purpose is two-fold relating to multivalued mapping satisfying condition [1]; first we have proved a localized version of a theorem due to ciric [2] and second, we have established a convergence theorem concerning fixed point.

Mathematics Subject Classification (1991) :
Primary - 54H25; Secondary - 47H10
Key words : Orbitally Complete Metric Space, Set Valued Mapping, Hausdorff Metric, Upper Semi Continuous Function, Fixed Point.

INTRODUCTION

The study of fixed point theorems concerning multifunctions has been developed extensively by many authors. The fixed point theorems for multifunctions are often applied to solve problems in connection with mathematical economics, optimization and optimal control [4, 8].

The developments of geometric fixed point theory for multifunctions were initiated by Nadler, Jr. [7] and subsequently pursued by several authors [1, 3, 4, 5, 6, 8, 11]. The analysis for multifunc-

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tion is primarily based on the concept of convergence of sets.

PRELIMINARIES

Let (M, d) be a metric space and let $C(M) = \{A \in 2^M : A$ is a closed subset of $M\}$.

Define the distance between ay two members E, F in $C(M)$ by

$$H(E, F) = \max\left\{\sup_{y \in E} D(y, F), \sup_{z \in F} D(z, E)\right\}$$

Where $D(x, A) = \inf_{y \in A} d(x, y)$, for any non-empty subset A of M. This H is called the Hausdorff metric in $C(M)$; it can be characterised by

$$H(E, F) = \sup_{x \in M} |D(x, E) - D(x, F)| \text{ and also}$$

$$H(E, F) = \inf\{r \mid E \subseteq F_r \text{ and } F \subseteq E_r\}$$

where $A_r = \{x \mid D(x, A) < r\}$ for $r > 0$.

We write $E_n \xrightarrow{H} E$ (as $n \to \infty$) or $E = H\text{-}\lim E_n$ for the sequence $\{E_n\}_n$ in $C(M)$ converging to E in $C(M)$, with respect to the Hausdorff metric H

i.e. $H(E_n, E) \to 0$ (as $n \to \infty$) or $E = \left(\bigcap_{n \geq 1}\right) cl\left(\bigcap_{k < n} E_k\right)$

If M is complete, then it can be shown that $(C(M), H)$ is a complete metric space.

Let X and Y be two sets. A multifunction $F : X \to 2^Y$ is a function from X into the power set 2^Y of Y, that is, for each x in X, Fx is a subset of Y.

For multifunction, F, the notion of fixed point has been modified. A point x in X is said to be a fixed point of F if and only if x is contained in its F-image i.e. $x \in Fx$.

A multifunction $F : M \to C(M)$ is called a multivalued contraction mapping if and only if there exists a fixed q. $0<q<1$, such that

$$H(Fx, Fy) \leq q \; d(x, y), \text{ for all } x, y \text{ in } M.$$

Recently, Ciric [2] proved fixed point theorems for multifunction F on M which are not necessarily continuous and which satisfy a condition of type,

(1) $\min\{H(Fx, Fy), D(x, Fx), D(y, Fy)\} - \min\{D(x, Fy), D(y, Fx)\} \leq q \; d(x, y)$ for some $0 < q < 1$ and for all x, y in M.

In doing so he generalised some results of Nadler [7], Dube and Singh [3] concerning multivalued contraction mappings. In order to establish these extension he introduced the notion of orbit of multifunction F and F-orbitally complete metric space.

An orbit of F at the point x in M is a sequence $\{x_n : x_n \in F x_{n-1}\}$.

A space M is said to be F - orbitally complete if and only if every Cauchy sequence of the form $\left\{x_{n_i} : x_{n_i} \in F x_{n_i - 1}\right\}$ converges in M.

A multifunction F is said to be upper semi-continuous if for any sequences $\{x_n\}_n$ and $\{y_n\}_n$ in M, $y_n \in F x_n$, the conditions $\lim x_n = x$ and $\lim y_n = y$ would imply $y \in Fx$.

F is said to be lower semi-continuous if for any sequence $\{x_n\}_n$ in M there exists a sequence $\{y_n\}_n$ with $y_n \in Fx_n$ such that the condition $\lim x_n = x$ and $\lim y_n = y$ would imply $y \in Fx$.

Finally, F is continuous if it is both lower and upper semi continuous.

A multifunction F is orbitally upper semi continuous, for any orbit $\{x_n\}$ of F at some x in M

$$x_n \to u^{\cdot} \in M \Rightarrow u \in F u.$$

The objective of this paper is two-fold. First, we prove a localized version of Theorem - 5 of Ciric [2] and secondly, we investigate a convergence theorem concerning fixed points satisfying condition [1] to multivalued mappings.

THEOREM 1. Let $B = B(x_0, r) = \{ x \in M : d(x_0, x) < r\}$ for any $x_0 \in M$ and $r > 0$ where (M, d) is an orbitally complete metric space. Let $f: B \rightarrow C(M)$ be an orbitally upper semi-continuous multifunction satisfying condition 1] for x , y in B and 2] $D(x_0, Fx_0) < (1 - q)r$. Then F has a fixed point.

PROOF. It follows from condition - 2] of the theorem that there exists x_1 in Fx_0 such that $d(x_0, x_1) < (1 - q)r$ and so $x_1 \in B$.

We may assume that $H(Fx_0, Fx_1) > 0$, otherwise if follows that $Fx_0 = Fx_1$ yielding that $x_1 \in Fx_1$ i.e. x_1 is a fixed point of F. Now from condition 1],

$$\min\{H(Fx_0, Fx_1), D(x_0, Fx_0), D(x_1, Fx_1)\} - \min\{D(x_0, Fx_1), D(x_1, Fx_0)\}$$

$$\leq qd(x_0, x_1).$$

which implies that $\min\{H(Fx_0, Fx_1), D(x_0, Fx_0), D(x_1, Fx_1)\} \leq qd(x_0, x_1)$.

Since $x_1 \in Fx_0$, therefore, $D(x_1, Fx_1) \leq H(Fx_0, Fx_1)$ whence it follows that

3] $\quad \min\{D(x_0, Fx_0), D(x_1, Fx_1)\} \leq qd(x_0, x_1)$.

If $\min\{D(x_0, Fx_0), D(x_1, Fx_1)\} = D(x_0, Fx_0)$ then from (3) it follows that

$$\{D(x_0, Fx_0) \leq qd(x_0, x_1).$$

Now choose a real number a with $0 < a < 1$ such that

$$q^{-a} D(x_0, Fx_0) = (1-q)r.$$

Then, for $x_1 \in Fx_0$ we have $d(x_0, x_1) < q^{-a} D(x_0, Fx_0) \leq q^{1-a} d(x_0, x_1)$

which is absurd as $q^{1-a} < 1$.

Thus, from 3] it follows that $D(x_1, Fx_1) \leq qd(x_0, x_1) < q(1-q)r$ which implies that there exists $x_2 \in Fx_1$ such that

$d(x_1, x_2) < q(1-q)r.$

Now, $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) < (1-q)r + q\,(1-q)r = (1+q)\,(1-q)r$

$$= (1-q^2)\,r < r$$

Hence $x_2 \in B$.

Let us assume that $d(x_0, x_n) < (1+q+q^2) + \ldots + q^{n-1})(1-q)r = (1-q^n)r < r$

and that $d(x_{n-1}, x_n), < q^{n-1}(1-q)r$ with $x_n \in Fx_{n-1}$.

Now, we have that

4] $\min\{H(Fx_{n-1}, Fx_n), D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\} - \min\{D(x_n, Fx_{n-1}),$

$D(x_{n-1}, Fx_n)\} \leq qd\,(x_{n-1}, x_n).$

Since $x_n \in Fx_{n-1}$, we have $D(x_n, Fx_n) \leq H(Fx_{n-1}, Fx_n)$ and

$D(x_n, Fx_{n-1}) = 0$. Therefore 4] reduces to

5] $\min\{D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\} \leq qd(x_{n-1}, x_n).$

If $\min\{D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n)\} = D(x_{n-1}, Fx_{n-1})$ then from 5],

$D(x_{n-1}, Fx_{n-1}) \leq qD(x_{n-1}, x_n)$

Let a be a real number with $0 < a < 1$ such that $d(x_{n-1}, x_n) < q^{-a}D\,(x_{n-1,} Fx_{n-1})$ whence it follows that $d(x_{n-1}, x_n) < q^{1-a}\,d(x_{n-1}, x_n)$

which is absurd since $q^{1-a} < 1$.

Thus, $D(x_n, Fx_n) \leq qd\,(x_{n-1}, x_n) < q^n\,(1-q)r$, which implies that there exists

$x_{n+1} \in Fx_n$ such that

$d(x_n, x_{n+1}) < q^n\,(1-q)r.$

$$\begin{aligned} \text{Again, } d(x_0, x_{n+1}) &\leq d(x_0, x_n) + d(x_n, x_{n+1}) \\ &< (1+q+q^2+\ldots+q^{n-1})(1-q)r + q^n(1-q)r \\ &= (1+q+q^2+\ldots+q^n)(1-q)r \\ &= (1-q^{n+1})r < r \end{aligned}$$

which yields that $x_{n+1} \in B$. Thus, every member of the sequence $\{x_n\}_n$ is contained in B.

Now, for any $p \in N$ [= the set of naturals],

$$d(x_n, x_{n+p}) \leq \sum_{i=1}^{p} d(x_{n+i-1}, x_{n+i}) < \sum_{i=1}^{p} q^{n+i-1}(1-q)r = q^n(1-q^p)r < q^n r \to 0$$

$$as\ n \to \infty.$$

This shows that the sequence $\{x_n\}_n$ is a Cauchy sequence. Since $x_{n+1} \in Fx_n$ (for $n \in N \cup \{0\}$) and M is orbitally complete, there is a point $u \in M$ such that $\lim_{n} x_n$ By the orbitally upper semicontinuity of F, we have $u \in Fu$.

Thus, u is a fixed point of F.

This completes the proof of the theorem.

Now, we shall consider a family of multifunctions satisfying condition 1] and in the next theorem, we shall prove the existence of common fixed point of a pair of such multifunctions.

Theorem 2: Let (M, d) be *a* complete metric space and let $F_i : M \to CB(M)$ $(i=1)$ be two u.c.s. multivalued mappings satisfying condition

$$1^*]\ \min\{H(F_1x, F_2y), D(x, F_1x), D(y, F_2y)\}$$

$$- \min\{D(x, F_2y), D(y, F_1x),\} \leq qd(x,y),$$

for all $x, y \in M$ where $CB(M)$ denotes the family of all closed bounded subsets of M. Then F_1 and F_2 have a common fixed point in M.

Proof Select $x_0 \in X$ and $x_1 \in F_1x_0$. Assume that $H(F_1x_0, F_2x_1) \neq 0$, otherwise F_1 and F_2 have a common fixed point. Let $a \in (0, 1)$ be arbitrary.

Then there exists $x_2 \in F_2x_1$ such that $d(x_1,x_2) < q^{-a}\ D(x_1,F_2x_1)$ and there exists $x_3 \in F_1x_2$ such that $d(x_2,x_3) < q^{-a}\ D(x_2,F_1x_2)$. Continuing in this way we can obtain a sequence $\{x_n\}_n$ in X with

$x_{2n+1} \in F_1x_{2n}$ and $x_{2n+2} \in F_2x_{2n+1}$ such that

5] $\quad d(x_{2n,}\, x_{2n+1}) < q^{-a}\ D(x_{2n}, F_1x_{2n})$ and

$$d(x_{2n+1,}\, x_{2n+2}) < q^{-a}\ D(x_{2n+1}, F_2x_{2n+1})\text{ , for all } n \in N \cup \{0\}.$$

Now, we have

$$\min\{H(F_1x_{2n},F_2x_{2n+1}), D(x_{2n},F_1x_{2n}), D(x_{2n+1},F_2x_{2n+1})\} -$$

$$\min\{D(x_{2n},F_2x_{2n+1}), D(x_{2n+1},F_1x_{2n})\} \leq qd(x_{2n}, x_{2n+1}).$$

Since $x_{2n+1} \in F_1x_{2n}$, it clearly follows that

$$D(x_{2n+1}, F_2x_{2n+1}), \leq H(F_1x_{2n}, F_2x_{2n+1}) \text{ and } D(x_{2n+1}, F_1x_{2n}) = 0 \text{ and}$$

hence we have

$$\min\{D(x_{2n},F_1x_{2n}), D(x_{2n+1},F_2x_{2n+1})\} \leq qd(x_{2n}, x_{2n+1}). \text{ i.e}$$

6] $\min\{q^{-a}D(x_{2n},F_1x_{2n}),\ q^{-a}\ D(x_{2n+1},F_2x_{2n+1})\} \leq q^{1-a}d(x_{2n}, x_{2n+1})$.

If $\min\{q^{-a}D(x_{2n},F_1x_{2n}),\ q^{-a}\ D(x_{2n+1},F_2x_{2n+1})\} = q^{-a}D(x_{2n}, F_1x_{2n})$.

then from 5] and 6] it follows that

$$d(x_{2n},x_{2n+1}) < q^{1-a}\ d(x_{2n},x_{2n+1}) \text{ which is absurd since } q^{1-a} < 1.$$

Therefore, $\quad q^{-a}\ D(x_{2n+1},F_2x_{2n+1}) < q^{1-a}\ d(x_{2n},x_{2n+1})$.

form 5] it follows that $d(x_{2n+1},x_{2n+2}) < q^{1-a}\ d(x_{2n}, x_{2n+1})$.

Repeating this argument $2n$-times, we obtain,

$$d(x_{2n+1}, x_{2n+2}) < q^{1-a} d(x_{2n}, x_{2n+1}) < \dots < (q^{1-a})^{2n+1} d(x_0, x_1).$$

Similarly, $d(x_{2n}, x_{2n+1}) < (q^{1-a})^{2n} d(x_0, x_1)$.

Since $q^{1-a} < 1$, by routine calculation it can be easily seen that the sequence $\{x_n\}$ is a Cauchy sequence and hence it will converge to some $u \in M$ i.e. $\lim_n x_n = u \cdot$

Since $x_{2n+1} \in f_1 x_{2n}$ and $\lim_n x_{2n+1} = u = \lim_n x_{2n}$, then the upper semicontinuity of F_1 yields $u \in F_1 u$ i.e. u is a fixed point of F_1. Similarly, it can be shown that $u \in F_2 u$.

Hence u is a common fixed point of F_1 and F_2.

Theorem 3: Let (M, d) be a complete metric space and E a compact subset of M. Let for each $n \in N$ $T_n : E \to CB(M)$ be continuous multifunction satisfying condition

$$\min\{H(T_n x, T_{n+1} Y), D(x, T_n x), D(y, T_{n+1} y)\} - \min\{D(x, T_{n+1} y), D(y, T_n x),\}$$

$$\leq qd(x,y),$$

for all $x, y \in E$ and for some $q \in (0, 1)$. Let $T : E \to CB(M)$.

If $\{T_n\}_n$ converges pointwise to T in the sense of the Hausdorff metric, then T has a fixed point. Indeed, Every cluster opint of the sequence $\{u_n\}_n$ of fixed point u_n of T_n is a fixed point of T.

Proof: By theorem 2, for each $n \in N$, T_n and T_{n+1} have a common fixed point, say u_n in E, i.e. $u_n \in T_n u_n$ and $u_n \in T_{n+1} u_n$.

Thus, we have a sequence $\{u_n\}_n$ of fixed points in E and E being compact, there exists a subsequence $\{u_{n_i}\}_i$ of $\{u_n\}_n$ which converges to

some $u \in E$.

Let $\varepsilon > 0$ and choose an integer $p > 0$ such that for all $i \geq p$

$$H(T_{n_i}, u, Tu) < \frac{\varepsilon}{3} \text{ and } d(u_{n_i}, u) < \frac{\varepsilon}{3}$$

Since for each $n \in N$, T_n being continuous, for all $i \geq p$,

$$H(T_{n_i} u_{n_i}, T_{n_i} u) < \frac{\varepsilon}{3}$$

Thus, we have

$$\begin{aligned} D(u, Tu) &\leq d(u, u_{n_i}) + D(u_{n_i}, Tu) \\ &\leq d(u, u_{n_i}) + H(T_{n_i} U_{n_i}, Tu) \\ &\leq d(u, u_{n_i}) + H(T_{n_i} U_{n_i}, T_{n_i} u) + H(T_{n_i} u, Tu) < \varepsilon \end{aligned}$$

Hence $D(u, Tu) = 0$ and hence $u \in \overline{Tu} = Tu$ as $Tu \in CB(M)$, i.e. u is a fixed point of T.

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 87-94

PROBLEM OF LINE EXPLOSION IN A GAS CLOUD

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(Received 12-08-96)

ABSTRACT

In the present paper similarity solutions have been developed describing the propagation of cylindrical shock in non-uniform atmosphere taking counter gas pressure and radiation heat flux into account. The total energy of the explosion is constant.

INTRODUCTION

Lin (1954) obtained numerically the solution for cylindrically symmetric flow. Ray (1957) duscussed the problems of point and line explosion and found an exact analytic solution. Analytic solutions in the three cases of plane, cylindrical and spherical flow have been noted by Sakurai (1955). Rogers (1958) has also studied the similarity solutions for three cases in uniform atmosphere. Similarity solutions describing the flow of a perfect gas behind cylindrical shock waves with radiation heat flux are investigated.

The problem of propagation of shock waves in a non-homogeneous medium is of great interest in exploring the effect of explosion in the stars and atmosphere of the earth.

The radiation pressure and ratiation energy have been ignored.

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The gas in the undisturbed field is assumed to be at rest. We have also assumed the gas to be gray and opaque and the shock to be transparent and isothermal.

The total energy of the expanding wave has been considered to remain constant. The solution, however, is only applicable to a gaseous medium where the undisturbed pressure falls as the inverse square of the distance from the line of explosion.

BASIC EQUATIONS AND BOUNDARY CONDITIONS

The equation of conservation of mass, momentum and energy behind the wave are

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + \frac{\rho u}{r} = 0 \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0 \tag{2.2}$$

$$\frac{\partial e}{\partial t} + u \frac{\partial e}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial t} + pu \frac{\partial}{\partial r}\left(\frac{1}{\rho}\right) + \frac{1}{\rho r} \frac{\partial}{\partial r}(rq) = 0 \tag{2.3}$$

where ρ is the density, p, the pressure, u, the radial velocity, q, the heat flux, t, the time and E, the internal energy.

For an ideal gas,

$$e = \frac{p}{(\gamma - 1)\rho}, \qquad p = \Gamma \rho T \tag{2.4}$$

where γ is the adiabatic gas index, T, the temperature, and Γ the gas constant.

Considering local thermodynamic equilibrium and taking Rosseland's diffusion approximation. We have

$$q = \frac{-c\mu}{3} \frac{\partial}{\partial r} (\sigma T^4) \tag{2.5}$$

Where $\frac{1}{4}\sigma c$ is the Stefan-Boltzmann constant, c, the velocity of light, and μ, the mean-free path of radiation, is a function of density and temperature.

Following Wand (1966), We take

$$\mu = \mu_0 \rho^{\alpha} T^{\beta} \tag{2.6}$$

μ, ρ and β are constants. The distrubance is headed by an isothermal shock and the conditions are

$$\rho_2 (v - u_2) = \rho_1 v = m_n \tag{2.7}$$

$$p_2 - p_1 = m_s u_2 \tag{2.8}$$

$$e_2 \frac{p_2}{\rho_2} + \frac{1}{2}(v - u_2)^2 - \frac{q_2}{m_s} = e_1 + \frac{p_1}{\rho_1} + \frac{1}{2} v^2 \tag{2.9}$$

$$T_2 = T_1 \tag{2.10}$$

Where subscripts 1 and 2 are for the regions just ahead and just behind of the surface, respectively and v denotes the shock velocity.

In front of the shock in the undisturbed gaseous medium, the dendity and pressure distributions are as given below.

$$p_1 = AR^n \qquad n < 0 \tag{2.11}$$

$$\rho_1 + BR^{\omega} \qquad -2 < \omega < 0 \tag{2.12}$$

Where R is the shock radions $A.B.$ n and ω areconstants.

Now let us consider solutions of the equations (2.7) - (2.10) in the form

$$u = \frac{r}{t} v(\eta)$$

$$\rho = r^{K} t^{\lambda} \Omega(\eta)$$

$$p = r^{K+2}\, t^{\lambda-2}\, p(\eta) \tag{2.13}$$

$$q = r^{K+3}\, t^{\lambda-3}\, F(\eta)$$

where $\eta = r^{a} + t^{b}$ and λ, K, a and b are constants (2.14)

The total energy of the disturbance per unit length is

$$E = 2\pi \int_{0}^{R} \left(\frac{1}{2} \rho u^{2} + \frac{p}{\gamma - 1} \right) r dr \tag{2.15}$$

In terms of variable η we get

$$E = \frac{2\pi}{a} \int_{\infty}^{\eta_0} \left(\frac{1}{2} \Omega v^{2} + \frac{p}{\gamma - 1} \right) \eta^{[(k+4)/a-1]t[\lambda-2-(a/b)(k+4)]} d\eta \tag{2.16}$$

where η_0 is the value of η at the shock front.

We take the shock surface to be given by η_0 = constant and velocity of the shock surface as

$$v = \frac{-b}{a} \frac{R}{t} \tag{2.17}$$

The total energy of disturbance within the shock surface at any time t is constant. This by (2.16) requires that

$$\lambda - 2 - \frac{b}{a}(K + 4) = 0 \tag{2.18}$$

Suppose the Mach mamber at the shock front be defined by

$$M^{2} = \frac{\rho_1 v^{2}}{r p_1}$$

Substitution of (2.13) in the equation of motion (2.1) to (2.6), shock conditions (2.7) to (2.10) and Equation (2.19) and after using the relation (2.18) we obtain that similarity conditions, following Singh and

Vishwakarma (1983), are compatible when

$$k = \omega, \qquad \lambda = 0, \qquad a = -(4+\omega), \qquad b = 2$$

$$n = -2, \qquad \beta = \frac{-5}{2}, \qquad and\ \alpha = \frac{\omega+1}{\omega} \tag{2.20}$$

Therefore, the pressure distribution becomes

$$p_1 = AR^{-2} \tag{2.21}$$

and Equations (2.1) to (2.3) and (2.5) are now sought into the forms

$$\frac{\Omega'(\eta)}{\Omega(\eta)} = \frac{\eta(4+\omega)v'(\eta) - (2+\omega)v(\eta)}{\eta[2-(4+\omega)v(\eta)]} \tag{2.22}$$

$$\frac{P'(\eta)}{P(\eta)} = \frac{\Omega(\eta)}{p(\eta)}\left[\frac{\eta v'(\eta)\{2-(4+\omega)v(\eta)\} + v(\eta)(v(\eta)-1)}{\eta(4+\omega)}\right] + \frac{2+\omega}{\eta(4+\omega)} \tag{2.23}$$

$$\frac{F'(\eta)}{F(\eta)} = \frac{P'(\eta)}{F(\eta)}\left[\frac{2-(4+\omega)v(\eta)}{(\gamma-1)(4+\omega)}\right] + \frac{P(\eta)}{F(\eta)}\left[\frac{(\omega+2\gamma+2)v(\eta) - \gamma(4+\omega)\eta v'(\eta) - 2}{\eta(\gamma-1)(4+\omega)}\right] + \frac{1}{\eta} \tag{2.24}$$

$$\frac{F(\eta)}{P(\eta)} = -N\frac{P(\eta)^{1/2}}{\Omega(\eta)\left(\frac{3}{2}-\alpha\right)}\left[2-(4+\omega)\eta\left\{\frac{P'(\eta)}{P(\eta)} - \frac{\Omega'(\eta)}{\Omega(\eta)}\right\}\right] \tag{2.25}$$

where $N = \dfrac{4\sigma c\mu_0}{3\Gamma^{2/3}A^{1-\alpha}}$

= a nondimensional radiation parameter (2.26)

$$\frac{1}{N}\frac{F(\eta)\Omega(\eta)^{\frac{1}{2}-\alpha}}{(p(\eta))^{\frac{1}{2}}}\left[2-(4+\omega)v(\eta)\right]-\frac{2P(\eta)}{\Omega(\eta)}v^{'}(\eta)$$

$$=\frac{\left[(4+\omega)v(\eta)+\omega\right]-\left[2-(4+\omega)v(\eta)\left\{v(\eta)(v(\eta)-1)\right\}\right]}{\eta\left[\left\{2-(4+\omega)v(\eta)^{2}\right\}-(4+\omega)^{2}\frac{P(\eta)}{\Omega(\eta)}\right]} \tag{2.27}$$

The approximation shock conditions are

$$v(\eta_0)=\frac{2}{(4+\omega)}\left[1-\frac{1}{\gamma M^2}\right] \tag{2.28}$$

$$\Omega(\eta_0) = \gamma M^2 \tag{2.29}$$

$$P(\eta_0)=\frac{4}{(4+\omega)^2} \tag{2.30}$$

$$F(\eta_0)=\frac{1}{2}\left(\frac{2}{4+\omega}\right)^3\left[\frac{1}{\gamma^2 M^4}-1\right] \tag{2.31}$$

Which are the initial value for our numerical calculation, where we assume that $\eta^0 = 1$

RESULTS

Differential equations are numerically solved by the Runge-Kutta Method and the solutions are presented in a convenient form as

$$\frac{u}{u_2}=\left(\frac{\eta_0}{\eta}\right)^{\frac{1}{4+\omega}}\frac{v(\eta)}{v(\eta_0)} \tag{3.1}$$

TABLE - I

First set.

(i) $\gamma = \frac{4}{3}$, $M^2 = 20$, $N = 10, \omega = -1.5$, $\lambda = \frac{1}{3}$

η	$\frac{u}{u_2}$	$\frac{\rho}{\rho_2}$	$\frac{p}{p_2}$	$\frac{q}{q_2}$
1.00	1.00	1.00	1.00	1.00
1.20	0.95	0.73	0.59	1.08
1.40	0.92	0.60	0.37	1.15
1.60	0.90	0.55	0.22	1.19
1.80	0.87	0.70	0.12	1.21

TABLE - II

Second set.

(ii) $\gamma = \frac{5}{3}$, $M^2 = 20$, $N = 100$, $\omega = -1.75$, $\lambda = \frac{3}{7}$

η	$\frac{u}{u_2}$	$\frac{\rho}{\rho_2}$	$\frac{p}{p_2}$	$\frac{q}{q_2}$
1.00	1.00	1.00	1.00	1.00
1.20	0.92	0.70	0.69	0.882
1.40	0.87	0.55	0.53	0.833
1.60	0.83	0.46	0.44	0.815
1.80	0.80	0.40	0.38	0.810
2.00	0.77	0.36	0.34	0.822
2.20	0.75	0.34	0.31	0.832

$$\frac{\rho}{\rho_2} = \left(\frac{\eta_0}{\eta}\right)^{\frac{\omega}{4+\omega}} \frac{\Omega(\eta)}{\Omega(\eta_0)}, \tag{3.2}$$

$$\frac{p}{p_2} = \left(\frac{\eta_0}{\eta}\right)^{\frac{2+\omega}{4+\omega}} \frac{P(\eta)}{P(\eta_0)}, \tag{3.3}$$

and
$$\frac{q_1}{q_2} = \left(\frac{\eta_0}{\eta}\right)^{\frac{3+\omega}{4+\omega}} \frac{F(\eta)}{F(\eta_0)}, \tag{3.4}$$

The numerical results for a certain choice of parameters are reproduced in the form of tables. We calculate our results for the following two set of parameters.

(i) $\gamma = \frac{4}{3}$, $M^2 = 20$, $N = 10$, $\omega = -1.5$, $\lambda = \frac{1}{3}$

(ii) $\gamma = \frac{5}{3}$, $M^2 = 20$, $N = 100$, $\omega = -1.75$, $\lambda = \frac{3}{7}$

Nature of the field variables may bee seen through the tables I and II. The IInd set of parameters are more effective on the flow variables rather than the Ist. The radiation parameter N affects the variation of density, pressure and radiation heat flux at its value increases.

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Journal of Natural & Physical Science Vol. 9-10 (1995-96) pp 95-100

GENRAL FIXED POINT THEOREMS FOR MAPPINGS IN A HAUSDORFF SPACE

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(Received on 6-07-96 after revision 10-03-97)

ABSTRACT

In this note we establish two general fixed point theorems for selfmaps of a Hausdorff space, and then show that the results of Popa (1983) and Chugh and Rani (1992) are special cases of these theorems.

Mathematic Subject Classifications (1991) : 47H10

Keywords and Phrases: Hausdorff spaces, fixed points, pairs of maps

Sehie Park [4], Troy Hicks and Rhoades [2,3] and Rhoades [6,7] have published papers which establish fixed point theorems, in metric spaces and d-complete spaces, using general principles. We have shown that these general principles then contain a number of fixed point theorems of other authors as special cases. It is the purpose of this note to establish two general fixed point theorems for mappings in a Hausdorff space, and then show that the results of Popa [5] and Chugh and Rani [1] are special cases of these theorems.

The orbit of a point x_0 is defined to be the set $U = \{x_0,\ Tx_0, \ldots\ldots T^n x_0, \ldots\ldots\}$. T will be called x_0-orbitally continuous if T is continuous on U.

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Theorem 1. Let T be a selfmap of a Hausdorff space X, $f: X \times X \to IR^+$, f continuous and such that

(a) $f(x, y) \neq 0$ for each $x \neq y$,

(b) $f(Tx, T^2x) \leq qf(x, Tx)$ for each $x \in X$, $x \neq Tx$, q satisfying $0<q<1$,

(c) there exists a point $x_0 \in X$ such that $\{T^n x_0\}$ has a convergent subsequence, and

(d) T is x_0-orbitally continuous.

Then T has a fixed point

Proof. Define $x_n = T^n x_0$. We may assume that $x_n \neq x_{n+1}$ for each n, since, otherwise, T has a fixed point.

From condition (b), $f(x_n, x_{n+1}) \leq qf(x_{n-1}, x_n) \leq \ldots\ldots \leq q^n f(x_0, x_1)$. Therefore $\{f(x_n, x_{n+1})\}$ is a monotone decreasing positive sequence with limit 0.

From (c) let $\{x_{nk}\}$ denote the convergent subsequence of $\{x_n\}$ and let z denote the limit. Using (d), $z = \lim_k x_{nk}$ and $Tz = T(\lim_k x_{nk}) = \lim_k T(x_{nk}) = \lim_k x_{nk+1}$. Therefore, using the continuity of f, $f(z, Tz) = f(\lim_k x_{nk}, \lim_k x_{nk+1}) = \lim_k f(x_{nk}, x_{nk+1}) = 0$, since $\lim f(x_n, x_{n+1}) = 0$. From (a), $z = Tz$.

Corollary 1. (Theorem 2 of Popa [5]) Let T be a continuous selfmap of a Hausdorff space, $f: X \times X \to IR^+$, f continuous and such that

(a) $f(x, y) \neq 0$ for each $x \neq y$,

(b) $f(Tx, Ty) \leq \dfrac{af(x,Tx)f(y,Ty)}{f(x,y)} + bf(x, y)$ for all $x \neq y$, $a, b \geq 0$, $a + b < 1$, and

(c) $f^2(x, y) \geq f(x, x)f(y, y)$ for each $x \neq y \in X$.

If, for some $x_0 \in X$, $\{T^n x_0\}$ has a convergent subsequence, then T has a unique fixed point.

Proof. In (b) set $y = Tx$ to obtain condition (b) of Theorem 1. Then, from Theorem 1, T has a fixed point. Condition (c) implies uniqueness.

Corollary 2. (Theorem 3 of Chugh and Rani [1]) Let T be a continuous selfmap of a Hausdorff space, $f: X \times X \to IR^+$, f continuous and such that

(a) $f(x, y) \neq 0$ for each $x \neq y$,

(b) $f(Tx, Ty) \leq \dfrac{\alpha f(y,Ty)[1+f(x,Tx)]}{1+f(x,y)} + \beta f(x, y)$ for all $x \neq y$,

$\alpha, \beta > 0$, $\alpha + \beta < 1$, and

(c) $f(x, y) \geq \dfrac{1+f(x,x)}{1+f(y,y)}$ for all $x, y \in X$.

If, for some $x_0 \in X$ the sequence $\{T^n x_0\}$ has a convergent subsequence, then T has a unique fixed point.

Proof. Set $y = Tx$ in (b) to obtain (b) of Theorem 1. Then T has a fixed point. Uniqueness follows from condition (c).

Theorem 2. Let T_1 and T_2 be selfmaps of a Hausdorff space, $f: X \times X \to IR^+$, f continuous and such that

(a) $f(x, y) = f(y, x)$ for each $x, y \in X$,

(b) $f(x, y) \neq 0$ for each $x \neq y$, $x, y \in X$,

(c) $f(T_1 x, T_2 y) \leq q f(x, y)$ for each $x \neq y$, $x, y \in \{x_i\}$,

$\{x_i \mid x_{2n+1} = T_1 x_{2n}, x_{2n+2} = T_2 x_{2n+1}\} \subset X$, satisfying either $x = T_2 y$ or $y = T_1 x$,

(d) there exists a point $x_0 \in X$ such that $\{x_n\}$, as defined in (b), has a convergent subsequence, and

(e) T_1 and T_2 are $\{x_{2n}\}$, $\{x_{2n+1}\}$, - orbitally continuous, respactively.

Then either T_1 or T_2 has a fixed point or T_1 and T_2 have a common fixed point.

Proof. Suppose that $x_n = x_{n+1}$ for some n. If n is even, then we have $x_{2n} = x_{2n+1} = T_1x_{2n}$, and x_{2n} is a fixed point of T_1. Suppose that n is odd. Then we have $x_{2n+1} = x_{2n+2} = T_2x_{2n+1}$, and x_{2n+1} is a fixed point of T_2.

Suppose that $x_n \neq x_{n+1}$ for each n. then, from (c) and (a), $f(x_{2n+1}, x_{2n+2}) = f(T_1x_{2n}, T_2x_{2n+1}) \leq qf(x_{2n}, x_{2n+1})$ and $f(x_{2n}, x_{2n+1}) = f(T_2x_{2n-1}, T_1x_{2n}) = f(T_1x_{2n}, T_2x_{2n-1}) \leq qf(x_{2n+1}, x_{2n}) = qf(x_{2n}, x_{2n+1})$, so that, for each n, $f(x_n, x_{n+1}) \leq qf(x_{n-1}, x_n)$, which implies that $f(x_n, x_{n+1}) \leq q^n f(x_0, x_1)$, and lim $f(x_n, x_{n+1}) = 0$.

From condition (d) $\{x_n\}$ has a convergent subsequence. This convergent subsequence has either a convergent subsequence consisting of terms with only even subscripts or a convergent subsequence consisting of terms with only odd subscripts. For definiteness, assume that the convergent subsequence consists of only even subscripts. For notational convenience, denote this subsequence by $\{x_{2m}\}$. The other case is proved similarly. Using condition (d), with $z = \lim x_{2m}$, condition (e) implies that $T_1z = T_1(\lim x_{2m}) = \lim T_1(x_{2m}) = \lim x_{2m+1}$. Therefore, since f is continuous,

$$f(z, T_1z) = f(\lim x_{2m}, \lim x_{2m+1}) = \lim f(x_{2m}, \lim x_{2m+1}) = 0,$$

which, from (b), implies that $z = T_1z$. In a similar nammer it can be shown that $z = T_2z$.

Corollary 3. (Theorem 3 of Popa [5]) Let T_1, T_2 be continuous selfmaps of a Hausdorff space $f : X \times X \rightarrow IR^+$, f continuous and such that

(a) $f(x, y) = f(y, x)$ for each $x, y \in X$,

(b) $f(x, y) \neq 0$ for each $x \neq y \in X$,

(c) $f(T_1x, T_2y) \leq \dfrac{af(x,T_1x)f(y,T_2)}{f(x,y)} + bf(x, y)$ for all $x \neq y \in X$, a, b ≥ 0, $a + b < 1$, and

(d) $f^2(x, y) \leq f(x, x)f(y, y)$ for all $x, y \in X$.

if, for some $x_0 \in X$, the sequence $\{x_n\}$ defined by $T_1x_{2n} = x_{2n+1}$, $T_2x_{2n+1} = x_{2n+2}$, has a convergent subsequence. Then either T_1 or T_2 has a fixed point or T_1 and T_2 have a unique common fixed point.

Proof. Set $y = T_1x$ in (c) to obtain (b) of Theorem 2. Then T_1 and T_2 have a common fixed point. Uniqueness follows from condition (d).

Remark 1. Note that the restriction on the subsequence is not needed.

Corollary 4. (Theorem 4 of Chugh and Rani [1]) Let T_1, T_2 be continuous selfmaps of a Hausdorff space $f : X \times X \rightarrow IR^+$, f continuous and such that

(a) $f(x, y) = f(y, x)$ for each $x, y \in X$,

(b) $f(x, y) \neq 0$ for each $x \neq y \in X$,

(c) $f(T_1x, T_2y) \leq \dfrac{\alpha f(y,T_2y)[1+f(x,T_1x)]}{1+f(x,y)} + \beta f(x, y)$ for all $x \neq y$, $\alpha, \beta > 0$, $\alpha + \beta < 1$, and

(d) $f(x, y) \geq \dfrac{1+f(x,x)}{1+f(y,y)}$ for all $x, y \in X$.

If, for some $x_0 \in X$, the sequence $\{x_n\}$ defined by $T_1x_{2n} = x_{2n+1}$, $T_2x_{2n+1} = x_{2n+2}$, has a convergent subsequence of the type $\{x_{(2p+1)n}\}$, p fixed, then either T_1 or T_2 has a fixed point or T_1 and T_2 have a unique fixed point.

Proof. Set $y = T_1x$ in (c). Then one obtains condition (b) of Theorem 2. Therefore T_1 and T_2 have a common fixed point. Uniqueness follows from (d). As in Corollary 3, the special nature of the convergent subsequence is not needed.

Remark 2. Theorem 3 of Popa [5] and Theorem 4 of Chugh and Rani [1] are incorrectly stated. If $x_n = x_{n+1}$ for some n, one cannot conclude that also $x_{n+1} = x_{n+2}$, and therefore x_n is a common fixed point of T_1 and T_2. The reason for this is that, even though we know, from (b), that $f(x, y) = 0$ implies $x = y$, the converse is not assumed to be true.

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फार्म – 4

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

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हस्ताक्षर
श्याम नारायण सिंह
कुलसचिव

Volume 11 (1997)

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका

JOURNAL OF NATURAL & PHYSICAL SCIENCES

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Journal of Nature & Physical Sciences

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Journal of Natural & Physical Sciences Vol. 11 (1997) 1-12

COMPATIBLE MAPPINGS AND COMMON FIXED POINT FOR FOUR MAPPINGS

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(Received 28.07.1995)

ABSTRACT

In this paper we prove some common fixed point theorems in metric spaces which extend the results of B.Fisher, G. Jungck, M.S. Khan and M. Imdad.

Mathematics subject Classification (1980) : 54H25
Keywords : Common fixed points, commuting mappings and compatible mappings.

INTRODUCTION

In the year 1976, G, Jungck [5] gave a concept of common fixed point theorem for commuting mappings, which generalizes the well-known Banach's fixed point theorem. Later on, this result was generalized and extended in various ways by many authors : for instance, K.M. Das and K.V. Naik [1] K. Iseki and Bijendra Singh [4], B. Fisher [2], S.P. Singh [14], etc. Recently B. Fisher [3], M.S. Khan and M. Imdad [9], S.M. Kang and Y.P. Kim [8] proved some common fixed point theorems of three and four commuting mappings respectively.

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On the other hand, S. Sessa [12] defined a concept of generalization of commutativity, which is called weak commutativity and gave the common fixed point theorem for weakly commuting mappings, which generalizes the result of K.M. Das and K.V. Naik [1]. Further, G. Jungck [6] introduced more generalized commutativity, called compatibility, which is more general than that of weak commutativity. The utility of compatibility in the context of fixed point theory was initially demonstrated in extending a theorem of S.Park and J.S. Bae [11]. In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converses are not necessarily true as in the examples of Section 1.

The purpose of this paper is to propose a generalization of some common fixed point theorems, which extend the results of B. Fisher [3]. G. Jungck [7], M.S. Khan and M.Imdad [9] by employing a functional inequality and compatible mappings instead of commuting mappings and also give some examples to illustrate our main theorems.

PRELIMINARIES

The following was introduced by S. Sessa [12].

Definition 1.1 : Mappings S and T from a metric space (X,d) into itself, are called weakly commuting mappings on X, if $d(STx, TSx) \leq d(Sx,Tx)$ for all x in X.

Carearly, commuting mappings are weakly commuting, but the converse is not necessarily true, given by the following example :

Example 1.1 : Let $X = [0,1]$ with the Euclidean metric d. Define S and $T : X \to X$ by

$$Sx = \frac{x}{2-x} \quad \text{and} \quad Tx = \frac{x}{2}$$

for all x in X. Then we have for any x in X,

$$\begin{aligned} d(STx,\, TSx) &= \frac{x}{4-2x} - \frac{x}{4-x} \\ &= \frac{x^2}{(4-2x)\,(4-x)} \\ &\leq \frac{x^2}{4-2x} = \frac{x}{(2-x)} - \frac{x}{2} \\ &= d(Sx, Tx) \end{aligned}$$

Thus S and T are weakly commuting mappings on X, but they are not commuting on X since

$$STx = \frac{x}{4-2x} > \frac{x}{4-x} = TSx$$

for any non-zero x in X.

The following was given by G. Jungck [6] and [7] respectively.

Definition 1.2 : Mappings S and T from a metric space (X,d) into itself are said to be compatible mappings on X, if $\lim_{m\to\infty} d(STs_m, TSx_m) = 0$, when $\{x_m\}$ is a sequence in X such that $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} Tx_m = x$ for some point x in X.

If S and T are compatible mappings on X, then $d(STx, TSx)=0$, when $d(Sx,Tx) = 0$ for some x in X.

Obviously, weakly commuting mappings are compatible, but the converse is not necessarily true as in the following example :

Example 1.2. : Let $x = (-\infty, \infty)$ with the Euclidean metric d. Define S and $T : X \to X$ by

$$Sx = x^2 \quad \text{and} \quad Tx = 2x - 1$$

for all x in X. Since

$$d(Sx_m.Tx_m) = |x_m - 1|^2 \to 0, \qquad \text{if } x_m \to 1$$

$$\lim_{m\to\infty} d(Tsx_m, STx_m) = \lim_{m\to\infty} 2|x_m - 1|^2 \to 0 \text{ as } x_m \to 1$$

Thus, S and T are compatible on X, but are not weakly commuting mappings on X, since
$d(STx, TSx) = 2 > 1 = d(Sx, Tx)$ for $x=0$ in X.

Using the following Lemma 1.1 and Lemma 1.2 for our main theorems, which were proved by G. Jungck [6] and [7], respectively :

Lemma 1.1 : Let S and T be compatible mappings from a metric space (X,d) into itself. Suppose that $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} Tx_m = x$ for some x in X. Then $\lim_{m\to\infty} TSx_m = Sx$, if S is continuous.

Now let P, Q, S and T be mappings from a metric space (X,d) into itself satisfying the following conditions :

$$S(x) \subset Q(x) \quad \text{and} \quad T(x) \subset P(x) \tag{1.1}$$

$$d(Sx,Ty) \le \alpha \frac{d(Qy,Ty)\ [1+d(Px,Sx)]}{[1+d(Px,Qy)]} + \beta d(Px,Qy) \tag{1.2}$$

for all x,y in X, where $\alpha, \beta \le 0$, $\alpha+\beta < 1$. Then for an arbitrary point x_0 in X, by (1.1), we choose a point x_1 in X such that $Qx_1=Sx_0$ and for this point x_1, there exists a point x_2 in X such that $Px_2=Tx_1$ and so on. Proceeding in the similar manner, we can define a sequence $\{y_m\}$ in X such that

$$y_{2m+1}= Qx_{2m+1} = Sx_{2m} \text{ and } y_{2m} = Px_{2m} = Tx_{2m-1} \qquad (1.3)$$

Lemma 1.2 : Let $P.Q.S.$ and T be mappings from a metric space (X,d) into itself satisfying the conditions (1.1) and (1.2). Then the sequence $\{y_n\}$ defined by (1.3) is a Cauchy sequence in X.

FIXED POINT THEOREMS

In this section we prove common fixed point theorems for compatible mappings using functional inequality in complete metric spaces, we have the following :

Theorem 2.1 : Let P,Q,S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions (1.1) and (1.2) Suppose that

One of P,Q,S and T is continuous. (2.1)

Paris S,P and T,Q are compatible on X. (2.2)

Then P,Q,S and T have a unique common fixed point in X.

Proof : Let $\{y_m\}$ be the sequence in X defined by (1.3). By Lemma 1.2 $\{y_m\}$ is a Cauchy sequence and hence it converges to some point u in X. Consequently, the subsequences $\{Sx_{2m}\}$, $\{Px_{2m}\}$, $\{Tx_{2m-1}\}$ and $\{Qx_{2m+1}\}$ also converge to u.

Now suppose that P is continuous. Since S and P are compatible on X, Lemma 1.1 gives that

P^2x_{2m} and $SPx_{2m} \to Pu$ as $m \to \infty$.

By (1.2). we obtain

$$d(SPx_{2m}, Tx_{2m-1}) \leq \alpha \frac{d(Qx_{2m-1}, Tx_{2m-1})\,[1+d(P^2x_{2m}, SPx_{2m})]}{[1+d(P^2x_{2m}, Qx_{2m-1})]}$$

$$+ \beta\, d\,(P^2x_{2m}\,, Qx_{2m-1}).$$

By letting $m \to \infty$. we have

$$d(Pu,u) \leq \alpha \frac{d(u,u)\,[1+d(Pu,Pu)]}{[1+d(Pu,u)]} + \beta\, d\,(Pu,u)$$

$$\leq \beta\, d\,(Pu,\, u).$$

which implies

$$(1-\beta)\, d\,(Pu,\, u) \leq 0\ .$$

so that $u = Pu$. By (1.2), we also obtain

$$d(Su,Tx_{2m-1}) \leq \alpha \frac{d(Qx_{2m-1},Tx_{2m-1})\,[1+d(Pu,Su)]}{[1+d\,(Pu,Qx_{2m-1})]}$$

$$+ \beta\, d\,(Pu,\, Qx_{2m-1}).$$

By letting m $\to \infty$. we have

$$d(Su,u) \leq \alpha \frac{d(u,u)\,[1+d(Pu,Su)]}{[1+d(Pu,u)]} + \beta\, d\,(Pu,u)$$

$$\leq \beta\, d\,(Pu,\, u).$$

so that $u = Su$. Since $S(x) \subset Q(x)$ and hence there exists a point v in x such that $u = Su = Qv$.

$$d(u,Tv) = d(Su.Tv)$$

$$\leq \alpha \quad \frac{d(Qv, Tv)\,[1+d(u,Su)]}{[1+d(u,Qv)]} + \beta\, d\,(u,Qv)$$

which implies that $u = Tv$. Since T and Q are compatible on x and $Qv = Tv = u$, $d(QTv, TQv) = \theta$ and hence $Qu = QTv = TQv = Tu$

Moreover, by (1.2), we obtain

$$d(u,Qu) = d(Su,Tu)$$

$$\leq \alpha \, \frac{d(Qu,Tu)\,[1+d(u,Su)]}{[1+d(u,Qu)]} + \beta\, d\,(u,Qu),$$

so that $u = Qu$. Therefore, u is a common fixed point of P,Q,S and T. Similarly, we can also complete the proof, when Q is continuous.

Next suppose that S is continuous. Since S and P are compatible on X, it follows from Lemma (1.1) that

$$S^2x_{2m} \text{ and } PSx_{2m} \rightarrow Su \text{ as } m \rightarrow \infty.$$

By (1.2), we have

$$d(S^2x_{2m}, Tx_{2m-1}) \leq \alpha \, \frac{d(Qx_{2m-1}, Tx_{2m-1})\,[1+d(PSx_{2m}\, S^2x_{2m})]}{[1+d(PSx_{2m}, Qx_{2m-1})]}$$

$$+ \beta\, d\,(PSx_{2m}, Qx_{2m-1}).$$

By letting $m \rightarrow \infty$, we obtain

$$d(Su,u) \leq \alpha \, \frac{d(u,u)\,[1+d(Su,Su)]}{[1+d(Su,u)]} + \beta\, d\,(Su,u)$$

i.e. $d(Su,u) \leq \beta\, d\,(Su, u)$,

which implies

$$(1-\beta)\, d(Su,u) \leq 0,$$

so that $u = Su$. Hence, there exists a point ω in X, such that $u = Su = Q\omega$.

$$d(S^2x_{2m}, T\omega) \leq \alpha \frac{d(Q\omega ,T\omega)\ [1+d(PSx_{2m}\ S^2x_{2m})]}{[1+d(PSx_{2m} ,Q\omega)]} + \beta\, d\, (PSx_{2m},Q\omega).$$

By letting $m \rightarrow\infty$, we have

$$d(Su,T\omega) \leq \alpha \frac{d(Q\omega,T\omega)\ [1+d(u,Su)]}{[1+d(u,Q\omega)]} + \beta\, d\, (u,Q\omega).$$

which implies that $u = T\omega$. Since T and Q are compatible on X and $Q\omega = T\omega = u$, $d\,(QT\omega, TQ\omega) = 0$ and hence $Qu = QT\omega = Tu$. Moreover by (1.2). we have

$$d(Sx_{2m} ,Tu) \leq \alpha \frac{d(Qu,Tu)\ [1+d(Px_{2m}\ Sx_{2m})]}{[1+d(PSx_{2m},\ Qx_{2m-1})]} + \beta\, d\, (Px_{2m},Qu).$$

By letting $m \rightarrow\infty$, we obtain

$$d(Su,Tu) \leq \alpha \frac{d(u,Tu)\ [1+d(u,Su)]}{[1+d\ (u,u)]} + \beta\, d\, (u,u),$$

which implies

$$(1-\alpha)\, d(u,Tu) \leq 0\ .$$

so that $u = Tu$. Since $T(x) \subset P(x)$. There exists a point z in x such that $u = Tu = Pz$.

$$d(Sz,u) = d(Sz,\ Tu)$$

$$\leq \alpha \frac{d(Qu,Tu)\ [1+d(Pz,Sz)]}{[1+d\ (Pz,Qu)]} + \beta\, d\, (Pz,Qu)$$

$$\leq \alpha \frac{d(u,u)\ [1+d(Pz,Sz)]}{[1+d\ (Pz,u)]} + \beta\, d\ (Pz,u)$$

$$\leq \beta\ (Pz,u),$$

so that $Sz = u$. Since S and P are compatible on X and $Sz = Pz = u$. $d(PSz, SPz) = 0$ and hence $Pu = SPz = SPz = Su$. Therefore u is a common fixed points of P, Q, S and T. Similarly, we can also complete the proof when T is continuous.

Finally, in order to prove the uniqueness of u, suppose that u and z. $u \neq z$, are common fixed points of P,Q,S and T. Then by (1.2), we obtain

$$d(u,z) = d(Su,Tz)$$

$$\leq \alpha \frac{d(Qz,Tz)\ [1+d(Pu,Su)]}{[1+d\ (Pu,Qz)]} + \beta\, d\ (Pu,Qz)$$

$$d(u,z)\ \leq \alpha \frac{d(z,z)\ [1+d(u,u)]}{[1+d\ (u,z)]} + \beta\, d\ (u,z),$$

$$\leq \beta\, d\ (u,z),$$

which implies

$$(1-\beta)\, d(u,z) \leq 0.$$

which is contraction. Hence $u = z$. This completes the proof.

The following corollary follows immediately from theorem 2.1.

Corollary 2.2. Let P, Q, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions (1.1), (2.1), (2.2) and

$$(2.3)\quad d(Sx,Ty) \leq \alpha \frac{d(Qy,Sx)\ [1+d\ (Px,Ty)]}{[1+d\ (Px,Qy)]} + \beta\, d\ (Px,Qy)$$

for all x, y in X, where $\alpha,\ \beta \leq 0,\ \alpha + \beta < 1$. Then P, Q, S and T have a unique common fixed point in X.

Corollary 2.3 : Let P, Q, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions (1.1) (2.1). (2.2) and

$$d(Sx,Ty) \leq \alpha \frac{d(Px,Ty)\,[1+d\,(Qy,Sx)]}{[1+d\,(Px,Qy)]} + \beta\, d\,(Px,Qy). \qquad (2.4)$$

for all x,y in X, where $\alpha,\ \beta \geq 0.\ \alpha+\beta < 1$. Then P, Q, S and T and T have a unique common fixed point in X.

Theorem 2.4 : Let P, Q, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions (2.1), (2.5) and (2.6) for some positive integers $s,\ t,\ p$ and q

$$S^s(x) \subset Q^q(x) \text{ and } T^t(x) \subset P^p(x) \qquad (2.5)$$

$$d(S^s x, T^t y) \leq \alpha \frac{d(Q^q y, T^t y)\,[1+d(P^p x, S^s x)]}{[1+d(P^p x, Q^q y)]} + \beta\, d(P^p x, Q^q y) \qquad (2.6)$$

for all x,y in X, where $\alpha,\ \beta \geq 0.\ \alpha+\beta < 1$. Suppose that

S and T are commuting with P and Q respectively. (2.7)

Then P, Q, S and T have a unique common fixed point in X.

Proof : Since S and T commute with P and Q, S^s and T^t also commute with P^p and Q^q respectively. Thus, by theorem 2.1, there exists a unique point u in X such that $u = P^p u = Q^q u = S^s u = T^t u$.

From this, we obtain $Su = S^s\,(Su) = P^p\,(Su)$. Therefore, Su is a common fixed point of S^s and P^p. Also, Tu is a common fixed point of T^t and Q^q. Let $x = Su$ and $y = Tu$ in (2.6). Then we obtain,

$$d(Su,Tu) = d(S^s x\ ,\ T^t y)$$

$$\leq \alpha \frac{d(Q^q y, T^t y)\ [1+d(P^p x, S^s x)]}{[1+d(P^p x, Q^q y)]} + \beta\, d(P^p x, Q^q y)$$

$$\leq \beta\, d(P^p x, Q^q y)$$

$$\leq \beta\, d(Su, Tu),$$

which implies that $Su = Tu$ and hence it is the common fixed point of P^p, Q^q, S^s and T^t. Further, we obtain $Pu = S^s(Pu) = P^p(Pu)$. Therefore, Pu is a common fixed point of S^s and P^p. Also Qu is a common ifxed point of T^t and Q^q. Let $x = Pu$ and $y = Qu$ in (2.6). then we obtain

$$d(Pu,\ Qu) = d(S^s x,\ T^t y)$$

$$\leq \alpha \frac{d(Q^q y, T^t y)\ [1+d(P^p x, S^s x)]}{[1+d(P^p x, Q^q y)]} + \beta\, d(P^p x, Q^q y)$$

$$\leq \beta\, d(Pu, Qu),$$

which implies that $Pu = Qu$ and hence it is common fixed point of P^p, Q^q, S^s and T^t. By uniqueness u in X, this shows that $u = Pu = Qu = Su = Tu$. This completes the proof.

Remarks : Theorem 2.1 generalizers the result of G. Jungck [7] by using any one continuous mapping as opposed to the continuity of both P and Q.

Theorem 2.1 and corollary 2.2 and 2.3 also generalize the result of B. Fisher [3] by employing compatible mappings instead of commuting mappings. Further the condition (1.2) is more general than the condition of B. Fisher [3]

Theorem 2.4 extends the result of M.S. Khan and M.Imdad [9 by increasing the number of mappings and using any one continuou mapping as opposed to the continuity of both P and Q.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 13-20

A COMMON FIXED POINT THEOREM

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(Received 26.02.1996)

ABSTRACT

The purpose of this paper is to obtain a common fixed point theorem which gives proper generalizations of recent results due to Jachymski; Pant, Joshi and Pande; and Rhoades, Park and Moon. Our reuslt applies to a much wider class of mappings than covered by known results of this type.

Mathematics Subject Classification : 54 H 25.
Keywords : Compatible mappings, fixed point theorems, contractive conditions, coincidence points.

INTRODUCTION

The study of common fixed points of contractive type mappings is presently an area of intense research activity. Most of the recent results deal with four mappings or a sequence of mappings. The most general common fixed point theorems for sequences of mappings are those due to Jachymski [2], Pant, Joshi and Pande [9] and Rhoades, Park and Moon [10]. The theorems concerning sequences of mappings generally require each mappings to satisfy a compatibility condition , a condition on its

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range and a strong type of contractive condition. In the present paper, we obtain a common fixed point theorem concerning a sequence of mappings under much weaker conditions. Thus, our theorem gives a proper generalization of known results in many respects.

Two selfmaps A and S of a metric space (X,d) are called compatible if $\lim_n d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X. This notion was introduced by Jungck [4]. It is well known that compatibility implies commutativity at coincidence points.

Let A_1, A_2, S and T be selfmappings of a set X such that $A_1X \subset TX$ and $A_2X \subset SX$. For x_0 in X, a sequence $\{y_n\}$ defined by $y_{2n} = A_1x_{2n} = Tx_{2n+1}$ and $y_{2n+1} = A_2x_{2n+1} = Sx_{2n+2}$ is called an S,T- iteration of x_0 under A_1 and A_2.

RESULTS

If $\{A_i\}$, $i=1,2,3,\ldots$, S and T be selfmappings of a metric space (X,d), in the sequel we shall denote

$$M_{1i}(x,y) = \max\ \{d(Sx,Ty),\ d(A_1x, Sx),\ d(A_1y, Ty),\ [d(A_1x, Ty)+d(A_1y,Sx)]/2\}.$$

Theorem : Let $\{A_i\}$, $i=1,2,3,\ldots$, S and T be selfmappings of a metric space (X,d) such that

(i) $A_1X \subset TX$ and $A_2X \subset SX$,

(ii) $d(A_1x, A_2y) \leq h\, M_{12}(x,y)$, $0 \leq h < 1$,

(iii) $d(A_1x, A_iy) < M_{1i}(x,y)$.

Let S be compatible with A_1 and T be compatible with A_i for some $i > 1$. Suppose that range of one of the mappings A_1, A_2, S or T be a complete subspace of X. Then all the A_i, S and T have a unique common fixed point.

Proof : Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by

$$y_{2n} = A_1 x_{2n} = T x_{2n+1},\ y_{2n+1} = A_2 x_{2n+1} = S x_{2n+2}.$$

This can be done by virtue of (i). Then, by virtue of (ii), we get

$$d(y_{2n}, y_{2n+1}) \le h\, d(y_{2n-1}, y_{2n})$$

and $d(y_{2n-1}, y_{2n}) \le h\, d(y_{2n-2}, y_{2n-1})$.

From these inequalities we infer that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Also, for any integer $p > 0$, we have

$$d(y_{2n+1}, y_{2(n+p)+2} \le h\, d(y_{2n}, Y_{2(n+p)+1} + h d(y_{2n}, y_{2n+1}).$$

Then $d(y_{2n}, y_{2(n+p)+1} \le d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, Y_{2(n+p)+2}) + d(y_{2(n+p)+2,}\, y_{2(n+p)+1})$

$$< h d(y_{2n}, Y_{2(n+p)+1}) + (2+h) d(y_{2n}, y_{2n+1}),$$

that is, $(1-h)\, d(y_{2n}, Y_{2(n+p)+1}) < (2+h)\, d(y_{2n}, y_{2n+1})$.

Similarly, $(1-h)\, d(y_{2n-1}, Y_{2(n+p)}) < (2+h)\, d(y_{2n}, y_{2n-1})$.

Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, the above inequalities yield

$$\lim_{n\to\infty} d(y_{2n}, y_{2(n+p)+1}) = 0 = \lim_{n\to\infty} d(y_{2n-1}, Y_{2(n+p)})$$

Hence $\{y_n\}$ is a Cauchy sequence.

Therefore, by our assumption, $\{y_n\}$ converges to a point in $SX \cup TX$. Let $y_n \to Su$ for some u in X. Then $y_{2n} = A_1 x_{2n} = T x_{2n+1} \to Su$ and $y_{2n+1} = A_2 x_{2n+1} = S x_{2n+2} \to Su$.

Now, using (ii) we get $d(A_1u, A_2x_{2n+1}) \le h\, M_{12}(u, x_{2n+1})$. On lettin $n \to \infty$ this inequality yields $d(A_1u, Su) \le h\, d(A_1u, Su)$, that is, $A_1u = Su$. Sinc the range of A_1 is contained in the range of T, there exists a point w in such that $A_1u = Tw$. We show that $Tw = A_iw$ for each $i > 1$. If $A_iw \ne Tw$ fo some $i > 1$, using (iii) we get

$$d(A_1u, A_iw) < M_{1i}(u, w) = d(A_1u, A_iw),$$

a contradiction. Hence $Su = A_1u = Tw = A_iw$ where $i > 1$.

Let T be compatible with A_j for some $j > 1$. Then, since compatibl maps commute at coincidence points, we get $A_jTw = TA_jw$, or equivalently, $A_jA_jw = A_jTw = TA_jw = TTw$. We assert that $A_jw = A_jA_jw =$ TA_jw. If $A_jw \ne A_jA_jw$, using (iii) we get

$$d(A_jw, A_jA_jw) = \mathrm{d}(A_1u, A_jA_jw) < M_{1j}(u, A_jw) = d(A_1u, A_jA_jw),$$

a contradiction. Thus $A_jw = A_1u$ is a common fixed point of A_j and T Similarly, since A_1 and S are compatible and since u is their coincidenc point, A_1u is a common fixed point of A_1 and S. Moreover, if $A_1u \ne A_iA_1$ for some $i > 1$, using (iii) we get

$$d(A_1u, A_iA_1u) = d(A_1u, A_iA_jw) < M_{1i}(u, A_jw) = d(A_1u, A_iA_1u),$$

a contradiction. Thus $A_1u = Su$ is a common fixed point of all the A_i, S an T. The proof is similar when it is assumed that $\{y_n\}$ converges to a poin in TX. Uniqueness of the common fixed point follows easily. This complete the proof of the theorem.

The following example illustrates our theorem.

Example : Let $X = [2,6]$ with usual metric d. Define mappings A_i, S, T $X \to X$, $i = 1, 2 \ldots$, by

$A_1x = 2$ for each x

$A_2x = 2$ if $x \le 3$ or > 4, $A_2x = x - 1$ if $3 < x \le 4$

$Sx = x$ if $x \leq 3$, $Sx = 4$ if $x > 3$

$Tx = 2$ if $x=2$ or > 4, $Tx=6$ if $2 < x \leq 3$, $Tx=3x-7$ if $3 < x \leq 4$

$A_3x = 2$ if $x \leq 7/3$ or > 3, $A_3x=(19/3)-x$ if $(7/3) < x \leq 3$

and for $i > 3$

$A_ix=2$ if $x \leq 2+(1/i)$ or > 4, $A_ix = 6+(1/i)- x$ if $2+(1/i) < x \leq 3$,

$A_i x = x-1$ if $3 < x \leq 4$.

Then $\{A_i\}$, S and T satisfy all the conditions of our theorem and have a unique common fixed point $x = 2$.

It may be observed in the above example that $A_1X \subset TX, A_2X \subset SX$, S is compatible with A_1 and T is compatible with A_3.

In view of the above example, we now compare our theorem with some well known results.

1. Our theorem is more general than Theorem 5.1 of Jachymski [2] in many respects. Jachymski's theorem assumes $A_iX \subset SX$ and A_i to be compatible with T for each $i \geq 2$. These strong conditions are neither required in our theorem nor satisfied in the above example. In our example, it is obvious that A_iX is not contained in SX when $i > 2$. To see that A_2 and T are noncompatible, let us consider a decreasing sequence $\{x_n\}$ in $X= [2,6]$ such that $3 < x_n < 10/3$ and $x_n \to 3$. Then $A_2x_n \to 2$, $Tx_n \to 2$, $A_2Tx_n \to 2$ and $TA_2x_n \to 6$. Hence A_2 and T are noncompatible. Similarly, by considering a decreasing sequence $\{x_n\}$ such that $x_n \to 3$ and $3 < x_n < 3+(1/3i)$, it can be shown that A_i and T are noncompatible when $i > 3$. It is also required in Jachymski's theorem that, for $i \geq 2$, A_1, A_i satisfy the contractive condition $d(A_1x,A_iy) \leq f_i(M_{1i}(x,y))$ where $f_i : R_+ \to R_+$ is an upper semicontinuous function such that $f_i(t) < t$ for each $t > 0$. In the above example, for every $i > 2$, A_1 and A_i donot satisfy such a contractive

condition since the required function $f_i(t)$ will not be upper semicontinuou at $t = 2$.

Similarly, our theorem is more general than Theorem 3.3 of Jachymsk [2] since that can be obtained as a special case of our theorem. Howeve in the above example, no pair of mappings A_1, A_i, $i \geq 2$, satisfies th conditions of Theorem 3.3 of [2].

II. Our theorem is more general than that due to Rhoades, Park an Moon [10] in following respects. The theorem of Rhoades et al require $A_i X \subset SX \cap TX$ and A_i to be compatible with both S and T for each valu of i. Our theorem assumes much weaker conditions than these. Moreove their result requires each A_i, A_j to satisfy an $(\in, h)$ - contractive conditio (a suitable correction to their $(\in, h)$ condition appeared in Jungck et al [5]), h being lower semicontinuous. However, in the above example, A A_i donot satisfy such a contractive condition for $i > 2$ since h will not b lower semicontinuous at $\in = 2$.

III. Likewise, the present theorem is more general than Theorem of Pant, Joshi and Pande [9] in two respects. In Theorem 2 of Pant et al i we take $\{P_i\} = \{A_1, A_1, A_1, \ldots.\}$ and $\{Q_i\} = \{A_2, A_3, \ldots.\}$ then A_1, A_i are required to satisfy an $(\in, \text{h})$ - contractive condition for each $i > 2$, h being nondecreasing. This condition is not satisfied in the above example since h is not found to be nondecreasing. Secondly, in Theorem 2 of Pant et a T is assumed to be compatible with a particular mapping while in the present theorem T may be compatible with any A_i for $i > 1$.

It is clear from the above discussion that our theorem applies to much wider class of mappings than covered by the above mentione results in [2], [9], [10]. Moreover, since other results of this type can be obtained as special cases of the results in [2], [9], [10], they can also be obtained as special cases of our theorem. Among the results that can be obtained as special cases of our theorem or can be generalized in the light of our theorem we mention those due to Fisher [1], Joshi and Pant [3], Meir and Keeler [6], Pant [7], [8], Sessa [11] and Tivari and Singh [12].

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Journal of Natural & Physical Sciences Vol. 11 (1997) 21-38

ON KANAN TYPE MAPS IN D-METRIC SPACES

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(Received 5.3.1996)

ABSTRACT

In this paper some fixed point theorems for the mappings in D-metric spaces satisfying certain contractive conditions are proved which include some well-known fixed point theorems of Banach [1], Kannan [10] and Maia [11] in ordinary metric spaces, and of Dhage [3,8] in D-metric spaces as the special cases.

Mathematics Subject Classifications : 47H10, 54H25
Keywords : D-metric space, contraction map, fixed point etc.

INTRODUCTION

The fixed point theory is one of the important topics that has been discussed in topological spaces and which has several important applications in the nonlinear analysis. In case of a metric space, the fixed point theory is culminated to its peak point and at present a considerable literature is available in this direction. In almost all fixed point theorems in metric spaces a sufficient condition is given in terms of contractive inequality which guarantees the existence of fixed point of the mappings in question. In this paper some sufficient conditions, like in ordinary metric

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spaces, are given to the mapings in D-metric spaces which guarantee the existence of the fixed point.

Recently the present author [2,3,4] introduced the notion of a D-metric space as follows.

Let X denote a non-empty set. A real function D on $X\ xXx\ X$ is said to be a D-metric on X if it satisfies the following properties :

$(\mathrm{M}_1)\ D(x,y,z) \geq 0$ for all $x,y,z \in X$, (nonnegetivity)

$(\mathrm{M}_2)\ D(x,y,z) = 0$ if and only if $x=y=z$, (coincidence)

$(\mathrm{M}_3)\ D(x,y,z) = D(x,y,z)= \ldots.$ (symmetry)

$(\mathrm{M}_4)\ D(x,y,z) \leq D(x,y;z)+(a,y,z)$

for all $x,y,z \in X$ (tetrahedral inequality).

The non-empty set X together with a D-metric D on X is called a D-metric space and it is denoted by (X,D). The details of a D-metric space and its generalizations upto n variables are given in Dhage [4]. Below we give some basic definitions which will be used in the sequel.

A sequence $\{x_n\}$ of points of a D-metric space X is called D-convergent or simply convergent and converges to a point x if $\lim_{m,n\to\infty} D(x_m,x_n,x)=0$.A Sequence $\{x_n\}$ of points of a D-metric space X is said to be *D-Cauchy* if $\lim_{m,n,p\to\infty} D(x_m,x_n,x_p)=0$.A *complete* D-metric space X is one in which every D-Cauchy sequence converges to a point in it. A D-metric space X is said to be bounded if there exists a constant $M>0$ such that $D(x,y,z) \leq M$ for all $x,y,z \in X$, and the constant M is called a *D-bound* of X.

It is shown in Dhage [4] that the D-metric D is a continuous function in the topology of D-metric convergence which is Hausdorff in nature.

Let $f : X \to X$, then an orbit of f at a point $x \in X$ is a set $O(x)$ in X given by

$$O(x) = \{x, fx, f^2x, \ldots\} \tag{1.1}$$

An orbit $O(x), x \in X$, of f in a D-metric space X is said to be bounded if there exists a constant K such that $D(u,v,w) \leq K$ for all $u,v,w \in O(x)$.

A D-metric space is called f-orbitally bounded if the for each $x \in X$, $O(x)$is bounded. It is clear that every bounded D-metric space is f-orbitally bounded, but the converse may not be true.

Again a D-metric space X is said to be *f-orbitally complete* if every D-Cauchy sequence $\{x_n\}$ in $O(x), x \in X$, converges to a point in X. Finally a mapping f: $X \to X$ is said to be *f-orbitally continuous* if $x_n \to u$, $\{x_n\} \subset O(x)$, $x \in X$, then $fx_n \to fu$

Kannan [10] proved the following interesting fixed point theorem in the ordinary metric spaces which is responsible for the huge development of the metric fixed point theory.

Theorem 1.1 : Let the mapping f on a complete metric space into iteself satisfy

$$d(fx,fy) \leq \alpha[d(x,f(x)+d(y,fy)] \tag{1.2}$$

for all $x,y \in X$ and $0 \leq \alpha < 1/2$. Then f has a unique fixed point.

The mappings satisfying (1.2) are called Kannan mappings and are different from well-known Banach mappings characterized by the inequality

$$d(fx,fy) \leq \alpha\, d(x,y) \tag{1.3}$$

for all $x,y \in X$ and $0 \leq \alpha < 1$.

In this paper we investigate some new classes of contractive mappings in D-metric spaces possessing the fixed point property. In the following section we prove the main results of this paper.

MAIN RESULTS

Before going to the main results we prove a useful lemma.

Lemma 2.1 : Let $\{x_n\}$ be a bounded sequence in a D-metric space X with D-bound K satisfying

$$D(x_n, x_{n+1}, x_m) \leq \alpha^n K \tag{2.1}$$

for all $m > n \in N$, where $0 \leq \alpha < 1$. Then $\{x_n\}$ is D-Cauchy.

Proof : From (2.1), it follows that

$$D(x_n, x_{n+1}, x_{n+p}) \leq \alpha^n K$$

and $D(x_n, x_{n+1}, x_{n+p+t}) \leq \alpha^n K$ (2.2

for all $p, t \in N$.

Now

$$\begin{aligned}
& D(x_n, x_{n+p}, x_{n+p+t}) \\
& \leq D(x_n, x_{n+1}, x_{n+p+t}) + D(x_n, x_{n+p}, x_{n+p+t}) \\
& \qquad + D(x_n, x_{n+p}, x_{n+p+t}) \\
& \leq D(x_n, x_{n+1}, x_{n+p}) + D(x_n, x_{n+1}, x_{n+p+t}) \\
& + D(x_{n+1}, x_{n+2}, x_{n+p}) + D(x_{n+1}, x_{n+2}, x_{n+p+t}) \\
& + \ldots\ldots \\
& \leq \sum_{k=1}^{n+p-1} \alpha^k(K) + \sum_{k=1}^{n+p+t-1} \alpha^k(K)
\end{aligned}$$

$$(\because D(x_i, x_j, x_k,) \leq K, \forall\, i, j, k)$$

$$\leq 2\left(\sum_{k=1}^{n+p+t} \alpha^k\right)K$$

$$= \frac{2\ \alpha^n}{1-\ \alpha}(K)$$

$$\to 0 \text{ as } n \to \infty \ .$$

This shows that $\{x_n\}$ is D-Cauchy and the proof of the lemma is complete.

As a direct consequences of Lemma 2.1, we obtain

Lemma 2.2 : let X be a f-orbitally bounded D-metric space satisfying

$$D(f^n x, f^{n+1} x, f^m x) \leq \alpha^n K \tag{2.3}$$

for all $m > n \in N$ and $x \in X$, where $0 \leq \alpha < 1$, and K is the D-bound of O(x) in X. Then $\{f^n x\}$ is D-Cauchy for all $x \in X$.

Theorem 2.1 : Let f be a self - map of a f-orbitally bounded and f-orbitally complete D-metric space X satisfying

$$D(fx,fy,fz) \leq \alpha \max \{D(x,y,z), D(x,fx,fy), D(y,fy,fz), D(z,fz,fx)\\ D(x,fx,z), D(y,fy,x), D(z,fz,y)\} \tag{2.4}$$

for all $x, y, z \in X$, and $0 \leq \alpha < 1$. then f has a unique fixed point.

Proof : Suppose that $x = x_0 \in X$ is an arbitrary point and consider the sequence $\{x_n\}$ in defined by

$$x_{n+1} = fx_n, n = 0, 1, 2, \ldots.. \tag{2.5}$$

If $x_r = x_{r+1}$ for some $r \in N$, then $u = x_r$ is fixed point of f. Therefore, we assume that $x_n \neq x_{n+1}$ for each $n = 0, 1, 2, \ldots$ We show that (x_n) is D-Cauchy sequence. Let $x = x_{n-1}$, $y = x_n$ and $z = x_{n+1}$, then by inequality (2,4), we obtain

$$D(x_n,x_{n+1},x_{n+2})$$
$$=D(fx_{n-1},f(x_n,fx_{n+1})$$
$$\leq \alpha \max \{D(x_{n-1},x_n,x_{n+1}), D(x_n,x_{n+1},x_{n+2})\}$$

Since $D(x_n,x_{n+1},x_{n+2}) \leq \alpha\, D(x_n,x_{n+1},x_{n+2})$, $\alpha < 1$, is not possible, we have

$$D(x_n,x_{n+1},x_{n+2}), \leq \alpha\, D(x_{n-1},x_n,x_{n+1}),$$

for all $n \in N$.

Therefore

$$D(x_n,x_{n+1},x_{n+2}) \leq \alpha\, D(x_{n-1},x_n,x_{n+1})$$
$$\leq \alpha^n D(x_0,x_1,x_2) \qquad (*)$$

Now for $m>1$, we have

$$D(x_1,x_2,x_m)$$
$$=D(fx_0,fx_1,fx_{m-1})$$
$$\leq \alpha \max \{D(x_0,x_1,x_2), D(x_0,x_1,x_{m-1}), D(x_2,x_m,x_{m-1})\}$$
$$\leq \alpha \max \{\alpha^0 K, \alpha^0 \max_{\substack{0 \leq a \leq m-1 \\ 1 \leq b \leq m}} D(x_a,x_b,x_{m-1})\}$$
$$\leq \alpha K.$$

Similarly for $m>2$, we have

$$\mathrm{D}(x_2,x_3,x_\mathrm{m})$$
$$= D(fx_1,fx_2,fx_{m-1})$$
$$= \alpha \max \{D(x_1,x_2,x_{m-1}), D(x_1,x_2,x_3), D(x_{m-1},x_m,x_2)\}$$
$$\leq \alpha \max \{\alpha K, D(x_{m-1},x_m,x_2)\} \qquad (**)$$

But

$$\begin{aligned} &D(x_{m-1},x_m,x_2) \\ &\quad = D(x_m,x_2,x_{m-1}) \\ &\quad = D(fx_{m-1},fx_1,fx_{m-2}) \\ &\quad = \alpha \max\{D(x_1,x_{m-1},x_{m-2}), D(x_1,x_2,x_{m-1}), D(x_2,x_{m-1},x_{m-2})\} \end{aligned}$$

Substituting this in (**), we obtain

$$D(x_2,x_3,x_{\mathrm{m}}) \leq \alpha \max\{\alpha K,\ \alpha \max_{\substack{0\leq a\leq m-1 \\ 1\leq b\leq m}} D(x_a,x_b,x_{m-2})\}$$

$$\leq \alpha^2 K$$

In general

$$\begin{aligned} &D(x_n,x_{n+1},x_m) \\ &\quad = D(fx_{n-1},fx_n,fx_{m-1}) \end{aligned}$$

$$\leq \alpha \max\{\alpha^{n-1} K,\ \alpha^{n-1} \max_{\substack{0\leq a\leq m-1 \\ 1\leq b\leq m}} D(x_a,x_b,x_{m-n})\}$$

$$\leq \alpha^{n-1} K$$

for all $m > n \in N$.

Now an application of Lemma 2.2 yields that $\{x_n\}$ is a D-Cauchy sequence in X. Since X is a complete D-metric space, there is a point, say $u \in X$ such that $\lim D(x_m, x_n, u)=0$, i.e. $x_n \to u$. But every D-metric space X is hausdroff in the topology of D-metric convergence and so such a point u is unique, see for details Dhage [4]. Now we show that u is a fixed point of f.

If $u \neq fu$, then $D(u,u,fu) \neq 0$, $D(u,fu,fu) \neq 0$ and

$$\begin{aligned}
D(u,u,fu) &= \lim_{n\to\infty} D(x_{n+1},x_{n+1},fu) \\
&= \lim_{n\to\infty} D(fx_n,fx_n,fu) \\
&\leq \lim_{n\to\infty} \max\{D(x_n,x_n,u), D(x_n,x_{n+1},x_{n+2}), \\
&\qquad D(x_n,x_{n+1},u), D(u,fu,x_{n+1}), \\
&\qquad D(x_n,x_{n+1},u), D(u,fu,x_n)\} \\
&= \max\{0, D(u,u,fu)\} \\
&= \alpha D(u,u,fu)
\end{aligned}$$

which is a contradtiction, since $\alpha < 1$. Hence $u = fu$, i.e. u is a fixed point of f.

To prove the uniquencess, let $v(\neq u)$ be another fixed point of f. Then by condition (2,4) we get

$$\begin{aligned}
D(u,u,v) &= D(fu,fu,fv) \\
&\leq \alpha \max\{D(u,u,v), D(u,u,v)\} \\
&= \alpha D(u,v,v)
\end{aligned} \tag{2.6}$$

since $D(u,v,v) \leq \alpha D(u,v,v)$, $\alpha < 1$, is not possible.

Similarly by interchanging the role of u and v it is shown that

$$D(u,v,v) \leq \alpha D(u,u,v). \tag{2.7}$$

From (2.6) and (2.7), it follows that $D(u,v,u) \leq \alpha D(u,u,v)$, $\alpha < 1$, which is a contradiction and hence $u=v$. Thus f has *a* unique fixed point and the proof of the theorem is complete.

As a consequence of Theorem 2.1 we obtain the following corollaries.

Corollary 2.1 : Let f be a self-mapping of an f-orbitally bounded and f-orbitally complete D-metric space X. If there exists a $p \in N$ such that

$$\begin{aligned} &D(f^p x, f^p y, f^p z) \\ &\quad \leq \alpha \max (D(x,y,z)\ D(x, f^p x, f^p y)\ D(y, f^p y, f^p z), D(z, f^p z, f^p x) \\ &\qquad D(x, f^p x, z)\ D(y, f^p y, x), D(x, f^p z, y)\} \end{aligned} \tag{2.8}$$

for all $x,y, z \in X$ and $0 \leq \alpha < 1$. Then f has a unique fixed point

Corollary 2.2 : (Dhage [2.3]) : Let f be a self-map of an f-orbitally bounded and f-orbitally complete D-metric space X satisfying

$$D(fx,fy,fz) \leq \alpha D(x,y,z) \tag{2.9}$$

for all $x,y,z \in X$ and $0 \leq \alpha < 1$. Then f has a unique fixed point

Corollary 2.3 : (See [9,12]) : Let f be a self-map of a complete metric space X satisfying

$$d(fx,fy) \leq \alpha \max \{d(x,y)\ d(x,fx), d(y,fy)\} \tag{2.10}$$

for all $x,y,z \in X$ and $0 \leq \alpha < 1$. Then f has a unique fixed point

Proof : Define a D-metric D on X by

$$D(x,y,z)=\max\{d(x,y)\,d(y,z),\,d(x,z)\} \tag{2.11}$$

for all $x,y,z \in X$ (See Dhage [3]).

Obviously the completeness of (X,d) implies the completeness and hence an f-orbitally completeness of the D-metric space (X,D). Next we show that the completness of (X,d) together with the condition (2.10) implies the f-orbital boundeness of (X,D). Now for any $x \in X$, consider a sequence $\{x_n\}$ of points of X defined by (2.5). Clearly the sequence $\{x_n\}$ is same as the orbit (x) of f at a point $x \in X$. Then for $x=x_{n-1}$ and $y=x_n$, condition (2.10), we get

$$d(x_n,x_{n+1}) \leq \alpha\, d(x_{n-1},x_n) \tag{2.12}$$

for all $n \in N$. The inequality (2.12) shows that the sequence $\{x_n\}$ is Cauchy, and since (X,d) is complete, it converges to a point in X. But we know that every convergent sequence is bounded so $\{x_n\}$ is bounded w.r.t.D. In view of (2.11), it follow that $\{x_n\}$ is bounded w.r.t.D Hence (X,D) is an f-orbitally bounded D-metric space. Finally for any $x,y,z \in X$, from (2.10) and (2.11), we have

$$\begin{aligned}
&D(fx,fy,fz)\\
&\quad = \max\{d(fx,fy),d(fy,fz),\,d(fz,fx)\}\\
&\quad \leq \alpha\max\{d(x,y),d(y,z),d(z,x),d(x,fx)d(y,fy),d(z,fz)\}\\
&\quad \leq \alpha\max\{D(x,y,z),\,D(fx,fy),\,D(y,fy,fz),\,D(z,fz,fx)\\
&\qquad\qquad D(x,fx,x),D(y,fy,x),\,D(z,fz,y)\}
\end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied and hence an application of it yields that f has a unique fixed point. This completes the proof.

Remark 2.1 : Since the class of mappings f given by (2.10) includes the class of Kannan's mappings given by (1.2), Theorem 1.1 follows directly from our Theorem 2.2 as the special case.

Maia [11] initiated the study of the fixed point theorems in a metric space with two metrics. Sometimes it is possible that a D-metric space may not be complete w.r.t. a D-metric, but is complete w.r.t. another D-metric. Below in the following we deal with such situation and prove some fixed point theorems in tri-metric spaces which include some well-known fixed point theorems in ordinary metric spaces on the lines of Maia [11] as the special cases.

Theorem 2.2 : Let X be a D-metric space with three D-metrics D, D_1 and D_2 and $f : X \rightarrow X$. Assume the following conditions hold in X.

(i) $D_2(x,y,z) \leq D_1(x,y,z) \leq D(x,y,z)$ for all $x,u=y,z \in X$,

(ii) X is f-orbitally bounded w.r.t.D.,

(iii) X is f-orbitally complete w.r.t.D_1.

(iv) f is satisfies (2.4) on X w.r.t. D.

(v) f is satisfies (2.4) on X w.r.t.D.

Then f has a unique fixed point.

Proof : Let $x = x_0 \in X$ be arbitrary and consider the sequence $\{x_n\}$ in X defined by (2.5). Then proceeding as in the proof of Theorem 2.1 with similar argument, it is shown that $\{x_n\}$ is a D-Cauchy sequence w.r.t. D.

In view of hypothesis (i), it follows that (X_n) is also D-Cauchy w.r.t.D Since X is f-orbitally complete w.r.t.D_1, there is a point $u \in X$ such tha

$$\lim_{m,n\to\infty} D_1(x_m, x_n, u) = 0.$$

Now $D_2 \leq D_1$, on X^3, therefore we get

$$\lim_{m,n\to\infty} D_2(x_m, x_n, u) \leq \lim_{m,n\to\infty} D_1(x_m, x_n, u) = 0.$$

Again the uniquencess of the fixed point u directly follows from condition (v). This completes the proof.

Corollary 2.4 : Let X be a D-metric space with three D-metrics D, D_1 and D_2 and $f : X \to X$. Assume all the conditions (i) - (v) of Theorem 2.2 hold with (v) replaced by (2.8). Then f has a unique fixed point.

Corollary 2.5 : Let X be a tri D-metric space with three D-metrics D, D_1 and D_2 and $f : X \to X$. Assume all the conditions (i) - (v) of Theorem 2.2 hold with (v) replaced by (2.9). Then f has a unique fixed point.

Corollary 2.6 : Let X be a tri metric space with metrics D, D_1 and D_2 and let $f : X \to X$. Assume all the following conditions hold in X.

(i) $d_2(x,y) \leq d_1(x,y) \leq d(x,y)$ for all $x, y \in X$,

(ii) X is f-orbitally complete w.r.t. d_1,

(iii) f is f-orbitally continuous w.r.t.d_2, and

(iv) f is satisfies (2.10) on X.

Then f has a unique fixed point.

Proof : Define the D-metrics D, D_1 and D_2 on X by

$$\left.\begin{aligned} D(x,y,z) &= \max\{d(x,y)\,d(y,z),\, D(z,x)\} \\ D_1(x,y,z) &= \max\{d_1(x,y)\,d_1(y,z),\, D_1(z,x)\} \\ \text{and}\quad D_2(x,y,z) &= \max\{d_2(x,y)\,d_2(y,z),\, D(z,x)\} \end{aligned}\right\} \qquad (2.13)$$

for $x,y,z \in X$. Then proceeding as in the proof of Corollary 2.3 with similar arguments it is shown that all the conditions of Theorem 2.2 are satisfied and hence an application of it yields the desired result. This completes the proof.

Corollary 2.7 : (Maia [11]) : Let X be a bi-metric space with two metrics d and δ and let $f : X \rightarrow X$. Assume all the following conditions hold in X.

(i) $\delta(x,y) \leq d(x,y)$ for all $x,y \in X$,

(ii) X is complete w.r.t. δ,

(iii) f is continuous w.r.t. δ. and

(iv) f is satisfies (1.3) on X w.r.t.D.

Then f has a unique fixed point.

Proof : The proof follows directly from Corollary 2.6 by letting $d_1 = d_2 = \delta$.

Finally we deal with a special class of mappings called contractive mappings on a D-metric space which is not necessarily complete and bounded and prove some results on fixed point.

Theorem 2.3 : Let f be a self-map of a D-metric space X satisfying

$$D(fx,fy,fz) < \max\{D(x,y,z), D(x,fx,fy), D(y,fy,fz), D(z,fz,fx) \\ D(x,fx,z), D(y,fy,x) D(z,fz,y)\} \tag{2.14}$$

for all $x, y, z \in X$ with $\max\{D(x,y,z), D(x,fx,fy), D(y,fy,fz), D(z,fz,fx)$ $D(f,fx,z)\ D(y,fy,x)\ D(z,fz,y)\} \ddagger 0$. Suppose further that the sequence $\{x_n\}$ in X defined by (2.5) has a convergent subsequence converging to a point $u \in X$. If f, f^2 and f^3 are continuous at u, then u is a unique fixed point of f.

Proof : Now for $x = x_0$, $y = x_1$ and $z = x_2$ from (2.14) we obtain

$$D(x_1,x_2,x_3) < D(x_0,x_1,x_2).$$

Proceeding in this way, by induction, we get

$$D(x_{n-1}, x_n, x_{n+1}) > D(x_n, x_{n+2}, x_{n+2}) \tag{2.15}$$

for each n, n-1,2,.....

Let $c_n = D(x_n,x_{n+1},x_{n+2})$, then from (2.15), we have

$$c_0 > c_1 > c_2 > \ldots\ldots > c_n > \ldots\ldots \tag{2.16}$$

This shows that $\{c_n\}$ is a strictly decreasing sequence of positive real numbers which is bounded and hence congergent. Thus there is a point c such that

$$\lim_{m,n\to\infty} D_2(x_n, x_{n+1}, x_{n+2}) = \lim_{n\to\infty} c_n = c.$$

Further every subsequence $(c_n\}_k$ of $\{c_n\}$ is strictly decreasing and converges tot he same limit point c.

In view of $c_{n+1} > c_n$ for all $n \in N$ it follows that

$c_{n_k+1} < c_{n_k}$ for all $k \in N$.

Therefore

$$\lim_{k\to\infty} c_{n_k+1} = c = \lim_{k\to\infty} c_{n_k} \tag{2.18}$$

Now suppose that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to the point u, i.e. $\lim\limits_{k\to\infty} x_{n_k} = u$. By continuity of f (see Dhage (7) we obtain

$$\lim_{k\to\infty} x_{n_k+1} = \lim_{k\to\infty} f(x_{n_k}) = f(\lim_{k\to\infty} x_{n_k}) = fu.$$

Similarly

$$\lim_{k\to\infty} x_{n_k+2} = f^2u \text{ and } \lim_{k\to\infty} x_{n_k+3} = f^3u.$$

Since the D-metric D is continuous, we have

$$\lim_{k\to\infty} D(x_{n_k}, x_{n_k+1}, x_{n_k+2}) = D(u, fu, f^2u)$$

and $$\lim_{k\to\infty} D(x_{n_k+1}, x_{n_k+2}, x_{n_k+3}) = D(fu, f^2u, f^3u).$$

But in view of (2.18), we get

$$D(fu, f^2u, f^3u) = D(u, fu, f^2u). \tag{2.19}$$

If $u \neq fu$, then $D(u, fu, f^2u) \neq 0$ and from (2.14) we obtain

$$D(fu, f^2u, f^3u) < D(u, fu, f^2u) \tag{2.20}$$

which is a contradiction to (2.19). Hence $u = fu$. Again the uniqueness of u follows from the condition (2.14). This completes the proof.

As a consequence of Theorem 2.3 we obtain the following corollaries.

Corollary 2.8 : (Dhage [3]) : Let f be a continuous self-map of a compact D-metric space X satisfying (2.14), then f has a unique fixed point.

Proof : Since every sequence in the compact D-metric space has a convergent subsequence, the proof follows by an application of Theorem 2.3.

Corollary 2.9 : (Dhage [8]) : Let f be a self-map of a D-metric space X satisfying

$$D(fx, fy, fx) < D(x, y, z) \tag{2.21}$$

for all $x, y, z, \in X$ with $D(x, y, z) \neq 0$. Further suppose that the sequence $\{x_n\}$ in X defined by (2.5) has a convergent subsequence congerging to a point $u \in X$. Then u is a unique fixed point of f.

Proof : Since the condition (2.21) implies the continuity of f on X (See Dhage [6]). The conclusion now follows by an application of Theorem 2.3.

Corollary 2.10 : Let f be a self-map of a metric space X satisfying

$$d(fx,fy) < \max\ \{d(x,y),\ d(x,fx), d(y,fy)\} \qquad (2.22)$$

for all $x,y \in X$ with $\max\ \{d(x,y), d(f,fx), d(y,fy)\} \neq 0$. Suppose the sequence $\{x_n\}$ in X defined by (2.5) has a congergent subsequence converging to a point $u \in X$. Further, if f, f^2 and f^3 are continuous at u, then u is a unique fixed point of f.

Proof : Define a D-metric D on X by (2.11). Then it is clear that condition (2.22) implies the condition (2.14). Also the convergence of the subsequence w.r.t. the metric d implies the convergence w.r.t. the D-metric D. It is shown in (Dhage [7]) that the continuity of a mapping at a point w.r.t.D implies the continuity of the mapping at the same point w.r.t. the D-metric D. Hence f, f^2 and f^3 are continuous at $u \in X$ w.r.t. the D-metric on X. Thus all the conditions of Theorem 2.3 are satisfied and so an application of it yields the desired result. The proof is complete.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 39-46

BALANCED ROW COLUMN DESIGNS

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(Received 30.5.96)

ABSTRACT

In this paper balanced designs eliminating heterogeneity in two directions with number of treatments less than the number of rows and/or columns have been studied.

Mathematics Subject Classification (1991) : 62K99

Keywords : Elementary treatment contrasts, orthogonality, incidence matrices, g-inverse, doubly centered matrix, pseudo-dispersion matrix, Youden square, normal equation.

INTRODUCTION

The simplest form of designs eliminating heterogeneity in two directions are the Latin squares. The use of Latin square is limited by the fact that the number of experimental units needed to conduct this with s treatments is s^2. which increases very rapidly as s increases. When the number of treatments are not euqal to the number of rows and that of the number of columns then a Latin square design is not feasible to plan. An important class of designs

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for such situations was introduced by Youden [8], known as Youden square designs, which are some what less restrictive than the Latin square designs. Some other alternatives to Latin Square designs were suggested by Shrikhande [7], Hoblyn *et.al* [4], Freeman [1,2] and Freeman [3]. In all these designs, the row versus column classification is orthogonal, while the row versus treatment or column versus treatment classification may be non-orthogonal.

Row-column designs have been classified as $O:PP$, $O:TT$, $O:TP$ etc. by Hoblyn *et al.* [4], where O stands for orthogonality, T for balance and P for partial balance. Since the major interest is in the treatment effects, classification of a design on the basis of the overall design rather than on the basis of component designs (row-wise or column-wise design) is more meaningful. In view of this, in this paper, we study row-column designs which are incomplete in row and/or column and are balanced for estimates of elementary treatment contrasts. Unfortunately, it has not been possible to obtain a method for constructing balanced row-column designs with number of treatments less than the number of rows and/or columns.

BALANCED ROW-COLUMN DESIGNS

Let us consider a row-column design in which $n=pq$ experimental units are arranged in p rows and q columns and v treatments are assigned to these pq units in such a manner that each treatment is replicated r times in the design. Let the i-th treatment appears in the j-th row n_{1ij} times and in the k-th column n_{2ik} times where $n_{1ij} = 0$, or 1 and $n_{2ik} = 0$ or 1. We further assume that $p \leq v$ and $q \leq v$. Let $N_1=((n_{1ij}))$ and $N_2=((n_{2ik}))$ denote the treatments versus rows and treatments versus columns incidence matrices respectively.

It is then known that under usual homoscedastic fixed effect model, the reduced normal equations for estimating treatment effects after eliminating the row and column effects come out as,

$$F\tau = Q$$

where $F=rI-N_1N_1'/q-N_2N_2'/p+(r/v)JJ'$ (2.1)

τ is the vector of treatment effects, Q is the column vector of adjusted treatment totals given by

$$Q=T-N_1R/q-N_2C/p+N_1JG/pq$$

T, R and C being the column vectors of treatment, row and column totals respectively. G is the grand total of the observations, I denotes the identity matrix of order v and $J=(1,....,1)'$. The rank of F is utmost $(v-1)$. We now prove the following results.

Theorem 2.1 : A design eliminating heterogeneity in two directions is balanced if and only if the F matrix of the design is given by,

$$F=\theta\,(I- v^{-1}\,JJ'),$$

where θ is some positive scalar.

Proof : A square matrix A or order n is called a doubly centered matrix if its row sums and column sums are zero (see e.g. Rao and Mitra [6] pp 181). If A is doubly centered of order n and rank (n-1), then the unique doubly-centered generalized inverse (g-inverse) of A is A^+, the Moore-Penrose inverse of A (see Rao and Mitra [6] pp 181-182).

It can easily be seen that F is a doubly centered matrix of order v. Since rank $(F) = v-1$, it follows that a solution of normal equation is

$$\hat{\tau} = F^+Q$$

where F^+ is More-Penrose inverse of F, Further F^+ is also doubly-centered.

The pseudo dispersion matrix of τ is given by $\sigma^2 F^+$ where σ^2 is pe observation variance. Using the fact that F^+ is also doubl centered, one can easily show that the average variance of all estimated elementary treatment contrasts is given by

$$Aver.\ Var = 2\sigma^2/H$$

where H is the harmonic mean of non-zero eigen values of F.

Let $v^*(\hat{\tau})$ $Cov^*(\hat{\tau}_i, \hat{\tau}_j)$ denote the pseudo-variance of estimate i-th treatment effect and the pseudo-covariance between th estimated i-th and j-th treatment effects respectively. Then,

$$\sum_{i=1}^{v} v^*(\hat{\tau}_i) = tr\,(F^+) = \sigma^2\,(v-1)/H \qquad (2.2$$

where tr stands for the trace of a matrix. Also,

$$var(\hat{\tau}_1-\hat{\tau}_2) + Var\,(\hat{\tau}_1-\hat{\tau}_3)+\ldots Var(\hat{\tau}_1-\hat{\tau}_v)=v.v^*(\hat{\tau}_1)+\sum_{i=1}^{v} v^*(\hat{\tau}_1) \qquad (2.3$$

and

$$var(\hat{\tau}_2-\hat{\tau}_1) + Var\,(\hat{\tau}_2-\hat{\tau}_3)+\ldots Var(\hat{\tau}_2-\hat{\tau}_v)=v.v^*(\hat{\tau}_2)+\sum_{i=1}^{v} v^*(\hat{\tau}_i) \qquad (2.4$$

The solutions (2.3) and (2.4) follow from the fact that F^+ is doubly centered.

Now, let the design be balanced, i.e. $Var\,(\hat{\tau}_i-\hat{\tau}_j)$ is constant independent of i and j. Then it follows from (2.3) an (2.4) that.

$$v^*(\hat{\tau}_1)=v^*(\hat{\tau}_2)$$

Thus, from (2.2) we have.

$$v^*(\hat{\tau}_i) = \sigma^2 (1-v^{-1})/H \quad \text{for } i=1,2\ldots,v.$$

Also, since the design is balanced,

$$Var(\hat{\tau}_i - \hat{\tau}_j) = 2\,\sigma^2/H$$

or, $v^*(\hat{\tau}_i) + v^*(\hat{\tau}_j) - 2\,Cov^*\,(\hat{\tau}_i, \hat{\tau}_j) = 2\,\sigma^2/H$, giving,

$$Cov^*(\hat{\tau}_i, \hat{\tau}_j) = -\,\sigma^2/vH, \text{ for } i = j = 1,2,\ldots,v.$$

Thus, if the design is balanced

$$F^+ = H^{-1}\,(I - v^{-1}JJ')$$

Since $(I-v^{-1}JJ')$ is symmetric and idempotent, it follows that

$$\mathrm{F} = H\,(I - v^{-1}\,JJ')$$

conversely, let $F = \theta\,(I-v^{-1}JJ')$

Then,

$$F^+ = \theta^{-1}\,(I - v^{-1}JJ')$$

Thus, $\quad v^*(\hat{\tau}_i) = \sigma^2\,\alpha$ for $i = 1, \ldots, v$, and

$$Cov^*\,(\hat{\tau}_i, \hat{\tau}_j) = \sigma^2\,\beta \text{ for } i \neq j = 1,2\ldots v$$

where α and β are constants that do not depend on i and j. Thus,

$$Var\,(\hat{\tau}_i - \hat{\tau}_j) = v^*\,(\hat{\tau}_i) + v^*\,(\hat{\tau}_j) - 2\,Cov^*\,(\hat{\tau}_i, \hat{\tau}_j)$$

is also a constant independent of i and j showing that the design is balanced. Hence the theorem. The result contained in the theorem is well known, but its proof as given above appears to be new.

Theorem 2.2 : A row-column design with v treatments, p rows, q columns and equal replication r of treatment is balanced if, and only if.

$$pN_1N_1' + qN_2N_2' = [r(p+q) - \mu I] + \mu JJ' \qquad (2.5$$

where $\mu (v-1) = r (2pq-p-q)$

Proof : If (2.5) holds, we have,

$$F=rI- N_1N_1'/q - N_2N_2'/p + (r/v) JJ'$$

$$= rI-(1/pq)[\{ r(p+q) - \mu\} I + \mu JJ'] + (r/v) JJ'$$

Since $vr = pq$ and $\mu (v-1) = r (2pq-p-q)$. Thus,

$$F= (\mu-r^2)n^{-1} (I-v^{-1} JJ')$$

showing that the design is balanced.

Conversely, let the design be balanced, i.e.,

$$F = \theta (I-v^{-1}JJ')$$

on equating the traces of the matrices on both sides, we have (using 2.1)

$$vr-vrq^{-1}-vrp^{-1} + r=\theta (v-1)$$

giving $\theta = vr (1-q^{-1}-p^{-1} + v^{-1}) / (v-1)$. Then,

$$rI - N_1N_1'/q - N_2N_2'/p + (r/v) JJ' = vr(1-q^{-1}-p^{-1}+v^{-1})(v-1)^{-1}(I-v^{-1}JJ')$$

which gives,

$$pN_1N_1'+qN_2N_2' = [r(p+q)-\mu] \mathrm{I} + \mu JJ'$$

where, $\mu (v-1) = r (2pq - p - q)$

completing the proof of the theorem.

Latin square and Youden square designs are trivial examples of balanced row-column designs. The following design with 9 treatments, 6 rows and 6 columns, reported by Kshirsagar [5] is also balanced.

1	5	6	7	8	9
2	7	4	6	5	3
3	4	2	1	9	8
6	8	9	4	2	7
7	2	1	9	3	5
8	1	5	3	6	4

Table below gives some values of p,q,v and r which satisfy the necessary conditions for a design to be balanced where p and q both are less than v.

v	p	q	r
15	9	10	6
16	4	12	3
21	7	15	5
25	10	10	4
27	9	12	4

However, since these values satisfy only the necessary conditions, namely $vr=pq$ and $\mu(v\text{-}1)=r\,(2pq\text{-}p\text{-}q)$, it is not known whether these designs can be constructed or not. The design reported by Kshirsagar [5] is possibly the only existent balanced row-column design available in the literature for $r \leq 10$ with $p \leq v$ and $q \leq v$.

ACKNOWLEDGEMENT

The help and guidance received from Dr. A. Dey, Senior Professor, IASRI, New Delhi, in pereparation of this paper is gratefully acknowledged.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 47-54

ON ROBUST EXPERIMENTAL DESIGNS WITH RANDOM EFFECT MODEL

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(Received 30.5.1996)

ABSTRACT

Gopalan and Dey [6] have developed a criterion, on the lines similar to one suggested by Box and Draper [3]) for characterising the robust experimental designs assuming a fixed effect model or model I (Eisenhart [5]). In this paper a criterion for characterising robust experimental designs assuming a random effect model or model II (Eisenhart [5]) has been developed and the robustness of the designs eliminating heterogeneity in one direction has been studied.

Mathematics Subject Classification (1991) : 62 K 10

Keywords : Random effect modelm, robustness, idempotent matrices, eigen values, BIB designs, equireplicate, PBIB designs, heterogeneity, dispersion matrix, normal equations, incidence matrix.

INTRODUCTION

Box [2] has enumerated a number of criteria for judging a good response surface design for fitting an empirical interpolation function $Y=X\beta + e$ where $Y=(y_1, y_2, \dots, y_n)'$ is an nx1 vector of response observations, X is an nxp matrix of predictor variable observations, β is a px1 vector of parameters to be estimated and e

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is an nxl vector of residuals. One of the criterion is that the design should be insensitive to the wild observations. Box and Draper [3] have derived an appropriate numerical measure of a design's desirability in relation to the insensitivity to wild observations. Box and Draper [3] have derived an appropriate numerical measure of a design's desirability in relation to the insensitivity to wild observations. Box and Draper [3] have shown that in order that the predicted response y at any given point be insensitive to the outlier, the quantity $r = \sum_{u} r^2_{uu}$ should be minimised, where r_{uu} is the u-th diagonal element of the matrix $R = X(X'X)^{-1} X$. The designs which minimise r are called *robust*.

EXPERIMENTAL DESIGN WITH RANDOM EFFECT MODEL

Let us consider a model with random effect

$$Y = \mu J + \pi\alpha + U\beta + e$$

where $Y = (y_1, \ldots, y_n)'$ is vector of response of the i-th treatment in j-th block, μ is the general mean effects, $\alpha = (\alpha_1, \ldots, \alpha_v)'$, is the vector of treatment effects, $\beta = (\beta_1, \ldots, \beta_b)'$ is the vector of block effects, π' is the treatments vs block incidence matrix, U is the observations vs block incidence matrix and $J = (1, \ldots, 1)'$. Here β is assumed to be the vector of random block effects normally distributed with mean vector O and dispersion matrix $I\,\sigma_b^2$ and e are independent and normally distributed with mean vector O and dispersion matrix $I\,\sigma^2$, I is an identity matrix. The above model may be rewritten as

$$Y = X\theta + e_1$$

where $e_1 = U\beta + e$, $X = [J : \pi_{n,v}]$ and $\theta = (\mu, \alpha)'$

we have,

$$D(e_1) = UU'\,\sigma_b^2 + I\sigma^2$$

$$= Z^{-1}\sigma^2 \text{ (say)}$$

where Z^{-1} is the variance-covariance matrix and is function of $\Gamma = \sigma^2_b/\sigma^2$ which is assumed to be fixed and known. Application of usual least square technique to estimate the vector of parameters yields the normal equation,

$$\theta = (X'Z^{-1}X)^{-}X'Z^{-1}Y$$

where $(X'Z^{-1}X)^{-}$ is a generalized inverse (g-inverse) of $X'Z^{-1}X$ satisfying
$X'Z^{-1}X(X'Z^{-1}X)^{-}X'Z^{-1}X=X'Z^{-1}X$. It is well known that in the absence of any outlier $R^2_0/(n-m)$ is an unbiased estimator of σ^2 where,

$$R^2_0 = Y'[Z^{-1}-Z^{-1}X(X'Z^{-1}X)^{-}X'Z^{-1}]Y$$

m = rank (X) and n is the number of observations.

When u-th observation has added to it a quantity c making it an outlier, the bias or discrepancy in estimating σ^2 through $R^2_0{}^*$ is $d_u = c^2 a_{uu}/(n-m)$ where R^2_{0*} is the residual sum of squares in the presence of the outlier and a_{uu} is the u-th diagonal element of the matrix

$$\begin{aligned} A &= Z^{-1}-Z^{-1}X(X'Z^{-1}X)^{-}X'Z^{-1} \\ &= Z^{-1}-D(Z^{-1}X'\hat{\theta}) \end{aligned}$$

where D stands for the dispersion matrix.

If it were equally likely that c could occur with any of the n observations, giving rise to $d_1,\ldots,d_n$, the average discrepancy would be

$$\bar{d} = c^2 \sum_u a_{uu}/n(n-m)$$

which obviously is not fixed, unlike in the case of the fixed effect model, but is design dependent through A. In order that no unduly large discrepancy in the estimator of σ^2 is caused by the outlier, the average discrepancy should be minimised. A design minimising the average discrepancy, in case of a random effect model, may be called *robust*. We have,

$$\bar{D}(Z^{-1}X'\hat{\theta})/\sigma^2 = Z^{-1}\pi' \, (\pi Z^{-1}\pi')^{-1}\pi Z^{-1}$$

For getting the average discrepancy we need

$$\begin{aligned}\text{trace } Z^{-1}\pi' \, (\pi Z^{-1}\pi')^{-1}\pi Z^{-1} &= \text{trace } (\pi Z^{-1}\pi')^{-1}\pi Z^{-1} Z^{-1}\pi' \\ &= \text{trace } (\pi Z^{-1}\pi')^{-1}\pi Z^{-2}\pi'\end{aligned}$$

We examine below, robustness of some block designs based on the criterion discussed above.

BLOCK DEISGNS : In case of a block design,

$$Z^{-1} = \text{diag}[aI+bJ,\ldots,aI+bJ]$$

and $\pi Z^{-1}\pi' = ar^d + bNN'$

so, $\text{trace } (\pi Z^{-1}\pi')^{-1}\pi Z^{-2}\pi' = \text{trace}[ar^d+bNN'][a^2 r^d + b\,(2a+kbNN')]$

where r^d is a diagonal matrix with elements of the replication vector r as the diagonal elements and N is the treatment vs block incidence matrix, a and b are constants and k is the block size. Following Bose and Mesner [1], NN' can be decomposed in terms of the eigen values and the corresponding idempotent matrices of the algebra generated by the association matrices. $NN' = \sum_{i=0}^{m} \theta_i L_i$ where θ_i's are the given values of the matrix NN' and L_i's are the corresponding idempotent matrices. Assuming the design equireplicate we have,

$$\text{trace } (\pi Z^{-1}\pi')^{-1}\pi\, Z^{-2}\pi' = \sum_{i=0}^{m} [a^2r+b(2a+bk)\theta_i]/(ar+b\theta_i)] \tag{2.1}$$

showing that the average discrepancy is not fixed, as in the case of fixed effect model, but design dependent through θ_i's. In order that the average discrepancy is minimum, the trace given in (2.1) should be maximum subject to the condition that the sum of all the eigen values is constant. So maximising (2.1) subject to the condition

that $\sum_{i=0}^{m} \theta_i = vr$, we get,

$$\theta_i = rb^{-1} + \sqrt{[r(b-c)/\phi]} \tag{2.2}$$

where $c = b\,(2a+kb)$ and ϕ is the Lagrangian multiplier.

(a) **BIB Designs** : Let us consider a BIB (v,b,r,k,λ) design . Here

$\theta_0 = rk$ and $\sum_{i=0}^{m} \theta_i = r(v-k)$. Using (2.2) we get,

$$\theta_i = r-\lambda$$

i.e. each eigen value of NN' of a BIB design, other than rk, should be $(r-\lambda)$ in order that the design is robust, a property which is always satisfied by a BIB design, showing that a BIB design under random effect model is robust.

(b) ***PBIB Designs with two associate classes :*** Let n_0 n_1 and n_2 be the multiplicities of the roots θ_0, θ_1 and θ_2 respectively of NN'. If the design is connected then θ_0 is a simple root and hence $n_0 = 1$ and other multiplicities satisfy

$$n_1+n_2 = v-1$$

and trace $(NN') = rk+n_1\theta_1+n_2\theta_2$

Maximising (2,1) subject to the condition that sum of eigen values is constant, yields

$$\theta_1 = rv/(v-1) \qquad (2.3)$$

So, a PBIB design assuming a random effect model may be characterised as robust if (2.3) is satisfied. This condition has been throughly checked for all the PBIB designs catalogued by Clatworthy [4] and only two PBIB designs with the following parameters :

(i) $v = 4, b = 12, r = 6, k = 2, \lambda_1 = 4, \lambda_2 = 1$

(ii) $v = 9, b = 24, r = 8, k = 3, \lambda_1 = 1, \lambda_2 = 3$

satisfy the condition (2.3). These designs are R4 and LS13 of Clatworthy [4]. The robustness criterion $\tau = \theta_1 - vr/(v-1)$ for some of the PBIB designs has been tabulated in the Annex. For robust designs the value of τ will be equal to zero

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ANNEX

Table showing the value of the robustness criterion (τ) for some of the PBIB designs

v	r	k	b	λ_1	λ_2	τ	v	r	k	b	λ_1	λ_2	τ
6	2	4	3	2	1	0.4	8	9	6	57	9	1	6.8
6	4	4	6	4	2	0.8	42	10	6	70	10	1	7.8
6	6	4	9	6	3	1.2	10	4	8	5	4	3	2.4
6	8	4	12	8	4	1.6	10	8	8	10	8	6	4.9
6	10	4	15	10	5	2.0	12	2	8	3	2	1	1.8
8	3	4	6	3	1	0.6	12	4	8	6	4	2	3.6
8	6	4	12	6	2	1.2	12	6	8	9	6	3	5.4
8	9	4	18	9	3	1.8	12	8	8	12	8	4	7.3
10	4	4	10	4	1	1.6	12	10	8	15	10	5	9.1
10	8	4	20	8	2	3.1	12	10	8	15	10	6	2.1
12	5	4	15	5	1	2.4	14	4	8	7	4	2	0.3
12	10	4	30	10	2	5.1	14	8	8	14	8	4	0.6
14	6	4	21	6	1	3.5	16	3	8	6	3	1	2.8
16	7	4	28	7	1	4.5	16	6	8	12	6	2	9.6
18	8	4	36	8	1	5.5	16	7	8	14	7	3	0.5
20	9	4	45	9	1	6.5	16	9	8	18	9	3	14.4
22	10	4	55	10	1	7.5	18	8	8	18	8	3	1.5
8	3	6	4	3	2	1.4	20	4	8	10	4	1	7.8
8	6	6	8	6	4	2.9	20	6	8	15	6	2	1.7
8	9	6	12	8	6	4.2	20	8	8	20	8	2	15.6
9	2	6	3	2	1	0.8	24	5	8	15	5	1	10.8
9	4	6	6	4	2	1.5	24	10	8	30	10	2	13.5
9	6	6	9	6	3	2.2	26	4	8	13	4	1	7.8
9	8	6	12	8	4	3.0	26	8	8	26	8	2	3.7
9	10	6	15	10	5	3.7	28	6	8	21	6	1	13.8
10	6	6	10	6	3	0.7	32	5	8	20	5	1	2.2
12	3	6	6	3	1	2.7	32	7	8	28	7	1	16.8
12	5	6	10	5	2	0.5	32	10	8	40	10	2	3.7
12	6	6	12	6	2	5.5	36	8	8	36	8	1	19.8
12	9	6	18	9	3	8.2	40	9	8	45	9	1	22.8
12	10	6	20	10	4	1.1	44	10	8	55	10	1	25.8
14	3	6	7	3	1	0.8	50	8	8	50	8	1	5.9
14	6	6	14	6	2	1.5	56	9	8	63	9	1	7.0
14	9	6	21	9	3	2.3	12	3	9	4	3	2	0.3
15	4	6	10	4	1	4.7	12	6	9	8	6	4	0.6
15	8	6	20	8	2	9.4	12	9	9	12	9	6	0.8
18	4	6	12	4	1	1.8	15	6	9	10	6	3	2.6
18	5	6	15	5	1	6.7	18	5	9	10	5	2	13.7
18	8	6	24	8	2	3.3	18	10	9	20	10	4	7.4
18	10	6	30	10	2	13.4	21	3	9	7	3	1	2.8
20	9	6	30	9	2	4.5	21	6	9	14	6	2	5.7
21	6	6	21	6	1	8.7	21	9	9	21	9	3	8.5
24	7	6	28	7	1	0.2	27	4	9	12	4	1	4.8
26	6	6	26	6	1	3.8	27	8	9	24	8	2	9.7
27	8	6	36	8	1	12.7	30	9	9	30	9	2	11.7
30	7	6	35	7	1	4.8	39	6	9	26	6	1	8.8
30	9	6	45	9	1	14.7	45	7	9	35	7	1	10.8
33	10	6	55	10	1	16.7	57	9	9	57	9	1	14.8

v	r	k	b	λ_1	λ_2	τ	v	r	k	b	λ_1	λ_2	τ
63	10	9	70	10	1	16.8	6	4	3	8	0	2	4.8
12	5	10	6	5	4	3.5	6	5	3	12	0	3	7.2
12	10	10	12	10	8	6.9	6	8	3	16	0	4	9.6
15	2	10	3	2	1	2.8	6	10	3	20	0	5	12.0
15	4	10	6	4	2	5.7	9	3	3	9	0	1	3.4
15	6	10	9	6	3	8.6	9	6	3	18	0	2	6.7
15	8	10	12	8	4	11.4	9	9	3	27	0	3	10.1
15	10	10	15	10	5	14.3	12	4	3	16	0	1	4.4
18	10	10	18	10	5	0.6	12	8	3	32	0	2	8.7
20	3	10	6	3	1	6.8	15	5	3	25	0	1	5.4
20	6	10	12	6	2	13.7	10	4	5	8	0	2	4.4
20	9	10	18	9	3	20.5	10	6	5	12	0	3	6.7
20	9	10	18	9	4	0.5	10	8	5	16	0	4	8.9
22	5	10	11	5	2	0.8	10	10	5	20	0	5	11.1
22	10	10	22	10	4	1.5	15	6	5	18	0	2	6.4
25	4	10	10	4	1	10.8	15	9	5	27	0	3	9.6
25	8	10	20	8	2	21.7	20	4	5	16	0	1	4.
30	5	10	15	5	1	14.8	20	8	5	32	0	2	8.4
30	10	10	30	10	2	29.7	25	5	5	25	0	1	5.
35	6	10	21	6	1	18.8	14	6	7	12	0	3	6.4
40	7	10	28	7	1	22.2	14	8	7	16	0	4	8.6
42	5	10	21	5	1	2.9	14	10	7	20	0	5	10.
42	10	10	42	10	2	5.8	21	6	7	18	0	2	6.
45	8	10	36	8	1	26.8	21	9	7	27	0	3	9.
50	6	10	30	6	1	3.9	28	8	7	32	0	2	8.
50	9	10	45	9	1	30.8	9	7	7	9	6	5	3.9
55	10	10	55	10	1	34.8	12	7	7	12	6	2	17.4
82	10	10	82	10	1	7.9	12	7	7	12	2	4	6.6
4	2	2	4	0	1	2.8	12	7	7	12	3	4	6.
4	4	2	8	0	2	5.3	14	7	7	14	6	3	0.
4	6	2	12	0	3	8.0	18	7	7	18	6	1	23.6
4	8	2	16	0	4	10.7	10	6	2	30	1	0	4
4	10	2	20	0	5	13.3	10	3	2	15	0	1	7.
6	3	2	9	0	1	3.6	10	3	3	10	1	0	5.
6	6	2	18	0	2	7.2	10	6	3	20	2	0	11.
6	9	2	27	0	3	10.8	15	4	3	20	1	0	7.
8	4	2	16	0	1	4.6	15	3	3	15	0	1	9.
8	8	2	32	0	2	9.1	10	3	5	6	1	2	8.
10	5	2	25	0	1	5.6	10	6	5	12	3	2	16
10	10	2	50	0	2	11.1	10	6	5	12	2	4	22
12	6	2	36	0	1	6.6	15	2	5	6	1	0	7.
14	7	2	49	0	1	7.5	15	4	5	12	2	0	15
16	8	2	64	0	1	8.5	10	7	7	10	5	4	1.
20	10	2	100	0	1	10.5	28	2	7	8	1	0	11
6	2	3	4	0	1	2.4							

Journal of Natural & Physical Sciences Vol. 11 (1997) 55-64

A BIVISCOSITY MODEL OF CONVECTIVE STABILITY FOR BLOOD FLOW BETWEEN PARALLEL PLANTES

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(Received 30.10.1991, Revised 12.11.1996)

ABSTRACT

An analysis has been carried out for establishing the stability criterion for a vertical layer of blood treated as a non-Newtonian fluid (biviscosity fluid) heated from below filled in a tube formed by infinite parallel plannes. For equilibrium, a constant vertical temperature gradient is maintained in the tube. The boundaries of the plastic core region, moving with constant velocity, are determined for different values of Rayleigh number. Stability boundaries are investigated. For stable equilibrium, critical value of Rayleigh number has been analysed. Unstable equilibrium exists only under finite perturbation velocities, whose value are different for different values of parameters *x*.

Keywords : Biviscosity, Rayleigh number, Convective stability

INTRODUCTION

Blood is a complex fluid. Little is known definitely about the effect of the temperature on human blood. From a physiological standpoint the ideal temperature for blood is 37^0C. However, blood can function with reduced ability to absorb oxygen and deabsorb carbon-dioxide in a range of temperatures from 30^0C to 45^0C. Blood can survive for a long period of time if frozen (-5^0C to -10^0C) and

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thawed in an appropriate manner. Above 45^{0}C, the proteins in the blood will start to precipitate out of the plasma and coagulate. Enzymes in the blood do not function well above 45^{0}C. Change of blood temperature of this degree is unlikely to occur within a body as the normal body temperature regulation system will active in attempt to reverse this effect. Many situations now exist in which the blood is removed from the body for processing (for example oxygenation, hemodialysis etc) and again returned to the body. While in the body, temperature control of blood becomes important in order to prevent damage. The information about the temperature field for blood flowing in a tube is therefore of practical significance.

It has been shown experimentally by Charm and Kurland [4] that the rheological behaviour of blood can be approximated by the Casson's relation strongly suggesting the non-Newtonian nature of blood flow, particularly at low rates of shear. The flow of blood in a circular tube assuming it to be a Casson fluid has been investigated by Jones [5]. Flow of Casson fluid in non circular ducts has been investigated by Batra and Koshy [3]. Natural convective flow characteristics for a Casson fluid in a vertical uniform heat flux duct have been investigated by Batra and Mohan [2]. Effect of non-uniform inlet velocity profile on the heat transfer in the entrance region of a Casson fluid between parallel plates has been considered by Batra and Koshy [1].

In this paper, an attempt has been made to establish the criteria for the stability analysis of a vertical layer of blood treated as a biviscosity fluid heated from below filled in a tube formed by infinite parallel planes kept at a distant 2d apart. In the tube a constant vertical temperature is maintained. The problem is investigated by considering the solution in the three regions consisting of two viscous flow regions separated by a plastic region moving with constant velocity.

FORMULATION OF THE PROBLEM

A layer of a non-Newtonian fluid filled in a tube formed by infinite parallel planes located at a distance 2d apart is heated from below. The origin is chosen at the middle of the tube and cartesian co-ordinate system (x, y) is used. The constitutive equation for blood considered as a biviscosity model is given by Nakamura, *M.* and *T.* Swada (1988) as

$$\left.\begin{aligned} \tau_{ij} &= 2(\mu_\beta + \frac{p_y}{\sqrt{2\pi}}\, e_{ij}) \qquad && \pi \geq \pi_c, \\ &= 2(\mu_\beta + \frac{p_y}{\sqrt{2\pi_c}}\, e_{ij}) \qquad && \pi < \pi_c, \end{aligned}\right\} \tag{1}$$

where $\beta = \mu_\beta \sqrt{2\pi_c}/\mathrm{p_y}$. (2)

Equations governing the steady laminar incompressible flow for a biviscous fluid are given by

$$\rho \,[\, u\frac{\partial u}{\partial x} + v\, \frac{\partial u}{\partial y}\,] = \rho\, X + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\ , \tag{3}$$

$$\rho \,[\, u\frac{\partial v}{\partial x} + v\, \frac{\partial v}{\partial y}\,] = \rho\, Y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}.\ , \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5}$$

$$\rho\, C_p\left[\, u\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y}\,\right] = k\left[\,\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\,\right] + \mu\, \Phi, \qquad (6$$

where u, v are velocity components and X, Y the body forces, c_p is the specific heat at constant pressure and K is therma conductivity, Φ is dissipation function given by

$$\Phi = \tau_{ij}\; e_{ij} = \left[\frac{\partial u}{\partial x}\right]^2 + \left[\frac{\partial v}{\partial y}\right]^2 + \frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]^2 \qquad (7$$

and

$$\tau_{xx} = 2\left[\mu_\beta + \frac{p_y}{g}\right]\frac{\partial u}{\partial x}, \quad \tau_{xy} = \left[\mu_\beta + \frac{p_y}{g}\right]\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right],$$

$$\tau_{xy} = 2\left[\mu_\beta + \frac{p_y}{g}\right]\frac{\partial v}{\partial y}, \qquad (8$$

when

$$\begin{aligned} g &= \sqrt{2\pi}, \quad \pi \geq, \pi_C \\ &= \sqrt{2\pi}_C, \quad \pi <, \pi_C \end{aligned}$$

The problem is investigated under the following assumptions :

(1) A constant vertical temperature gradient is maintained.

(2) In equilibrium, there is steady laminar incompressible flow of the fluid.

(3) Heat produced by viscous dissipation and conduction negligible.

(4) For relatively low rates of flow, pressure is a function of y alone.

(5) Horizontal velocity component and vertical velocity gradient are negligible.

Under the above assumptions equations (3) and (5) are identically satisfied and equations (1), (4) and (6) reduce to

$$\tau = [\mu_\beta + \frac{p_y}{\sqrt{2\pi}}] \frac{dv}{dx} , \tag{9}$$

$$\frac{1}{p} \frac{dp}{dy} = -g + \frac{1}{\rho} \frac{\partial \tau_{xy}}{\partial x} \tag{10}$$

$$k \frac{\partial^2 T}{\partial x^2} = - \rho D c_p v \tag{11}$$

The equations are to be solved under the following boundary conditions :

$$v(\pm d) = 0. \tag{12}$$

$$T(\pm d) = 0. \tag{13}$$

also the equation must satisfy the close flow condition given by

$$\int_{-d}^{d} v \, d \, x = 0. \tag{14}$$

SOLUTION OF THE PROBLEM

If ρm is the density at mean fluid temperature T_m, then we have

$$k \frac{dp}{dy} = - \rho_m g . \tag{15}$$

Equation of state for small temperature difference is

$$\rho_m = \rho [1 - \gamma (T_m - T)] . \tag{16}$$

where γ is the coefficient of thermal expansion. Substituting (15) in equations (10) and equation (16) in equation (8), we get

$$\frac{1}{\rho}\frac{d\tau}{dx} + g\,\gamma\,\theta = 0 \quad , \qquad (17)$$

$$\alpha_1\,\theta'' = -\,DV \quad , \qquad (18$$

where $\theta = T - T_m$, $\alpha_1 = \dfrac{k}{\rho c_p}$ is thermal diffusivity. Non-dimensionalising equation (9), (17) and (18), we get

$$\tau' = (1+\beta^{-1}) + \frac{dv'}{dx} \,, \qquad (19$$

$$\frac{d\tau'}{dx'} + R\,\theta = 0 \qquad (20$$

$$\frac{d^2\theta'}{dx'^2} = -v' \qquad (21)$$

where

$$x' = \frac{x}{d}\,,\ v' = \frac{vd}{\alpha_1},\ \theta' = \frac{\theta}{Dd},\ = \frac{\tau d^2}{\mu_\beta \alpha_1} \quad ,$$

$$R = \frac{\rho g\ r\, d^4 D}{\alpha_1 \mu_\beta}.$$

where (12), (13) and (14) in dimensionless form become

$$v'(\pm 1) = 0 \quad , \qquad (22$$

$$\theta'\,(\pm 1) = 0 \quad , \qquad (23$$

$$\int_{-1}^{1} v'\, d\,x' = 0. \qquad (24$$

Eliminating τ' from equation (19) and (20) we get

$$\frac{d}{dx}\left[(1+\beta^{-1}) + \frac{dv'}{dx}\right] + R\,\theta' = 0,$$

$$(1+\beta^{-1})\,\frac{d^2v'}{dx'^2} = -\,R\,\theta' \qquad (25$$

Again eliminating v' from equations (21) and (25) we get

$$[(1+\beta^{-1}) \frac{d^4\theta'}{dx'^4} = R\,\theta' \ . \tag{26}$$

Solving the differential equation (26), we get

$$\theta' = c_1\, e^{\alpha x'} + c_2 e^{-\alpha x'} + c_3 \cos \alpha\, x' + c_4 \sin \alpha\, x' , \tag{27}$$

where, $\alpha = \dfrac{R}{1+\beta^{-1}}$,

Substitute the value of θ' in (21) we get v'

$$v' = \frac{d^2\theta'}{dx'^2} = -(\alpha^2 e^{\alpha x'} + \alpha^2 e^{-\alpha x'} - c_3\,\alpha^2 \cos \alpha x' - c_4\,\alpha^2 \sin \alpha x') \tag{28}$$

where c_1, c_2, c_3 and c_4 are constants.

Assuming $c_4 = \lambda$ (constant), for a particular model we have solved the simultaneous equations (27) and (28) under the boundary conditions (22) to (24) for many solutions of v' and θ' for different values of λ. The constants c_1, c_2 and c_3 have been obtained as

$$c_1 = -\frac{\lambda \sin \alpha \{(e^{\alpha} - e^{-\alpha})(\sin \alpha + \cos \alpha) + 2 \sin \alpha\, e^{-\alpha}}{\cos \lambda\, (e^{\lambda} - e^{-\lambda})^2 + \sin \lambda\, (e^{2\lambda} - e^{2\lambda})} , \tag{29}$$

$$c_2 = \lambda \sin \alpha \left[\frac{1}{(e^{\alpha} - e^{u})} - \frac{\{(e - e^{\alpha})^{-\alpha} (\sin \alpha + \cos \alpha) + 2 \sin \alpha\, e^{-2\alpha}}{\cos \alpha\, (e^{\alpha} - e^{-\alpha})^2 + \sin \alpha\, (e^{\alpha}(e^{\alpha} - e^{-\alpha}) + 1 - e^{-2\alpha})}\right], \tag{30}$$

$$c_3 = \frac{(C_1 + C_2)(e^{\alpha} - e^{-\alpha})}{2 \sin \alpha}, \tag{31}$$

RESULT AND DISCUSSION

On the basis of foregoing analysis and numerical calculations the following conclusion can be drawn for different models of blood. Velocity and temperature profiles have been shown for different values of Rayleigh number and biviscous parameter λ, which is responsible for change i nature of velocity and temperature profiles as compared with power la model Katiyar et. al [6].

1. From figs (1) and (2) is is evident that the velocity increases for increasing values of R for different values of x, the width of planes the velocity changes significantly for increasing values of th biviscous parameter, which shows the better efficiency of the mode as compared with other models as was expected.

2. Figs. (3) and (4) show that the temperature changes initially with x, but attains uniformity for all width for value of biviscou parameter. There is a uniform delay in temperature regulation syste when biviscous parameter increases which is physiologically tru for biothermal phenomenon in circulatory system.

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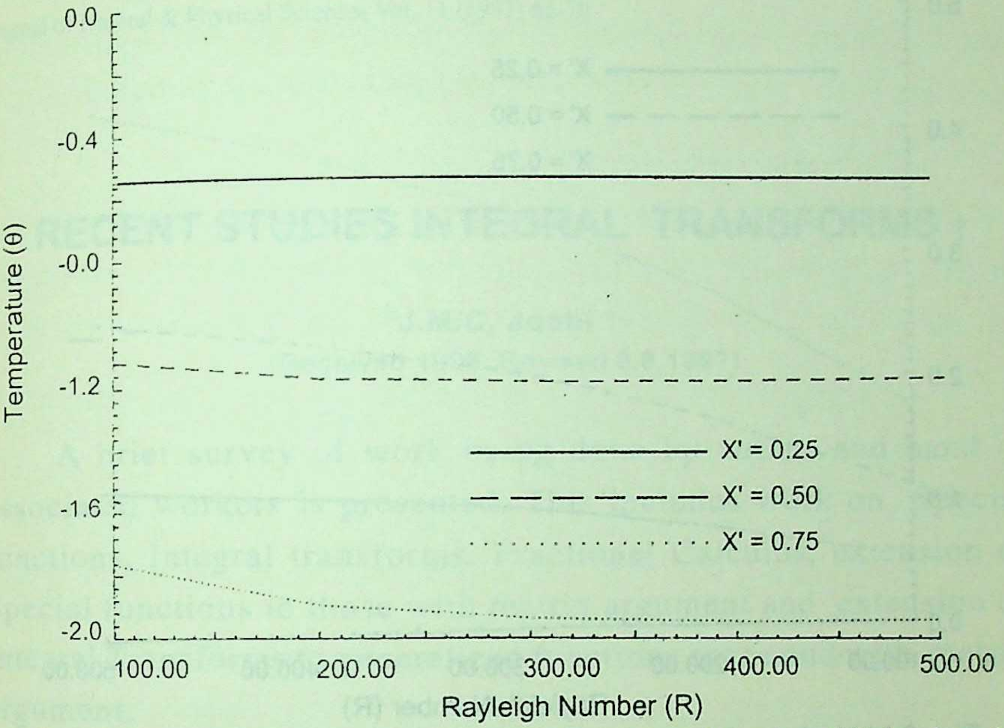

Fig. 1 Variation of Temperature with Rayleigh Number (R) for λ = 1.0

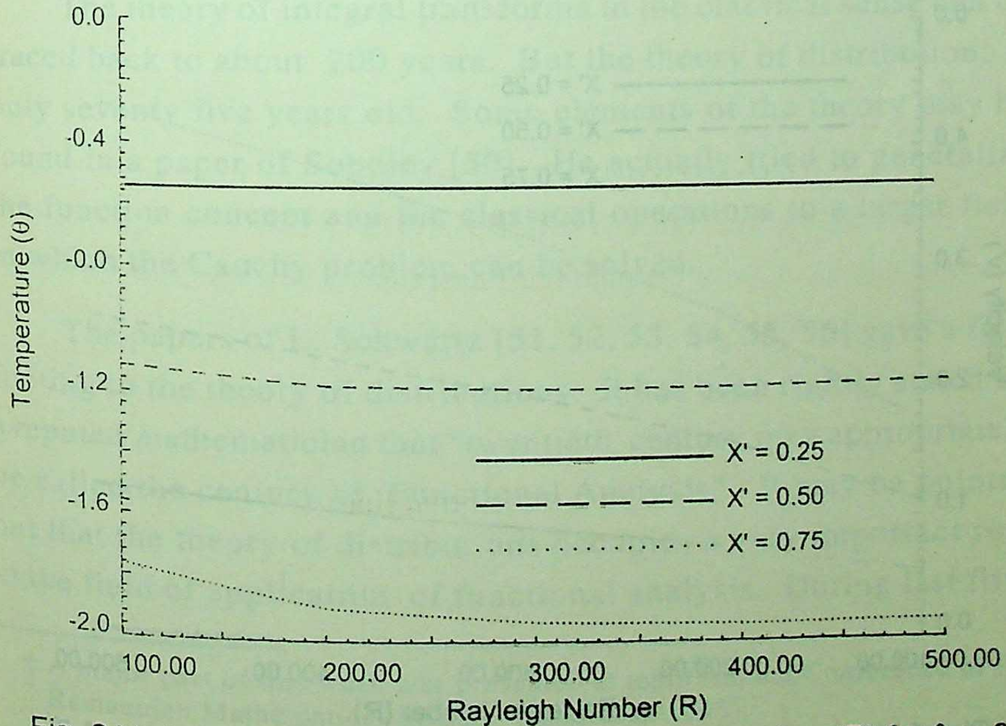

Fig. 2 Variation of Temperature with Rayleigh Number (R) for λ = 3.0

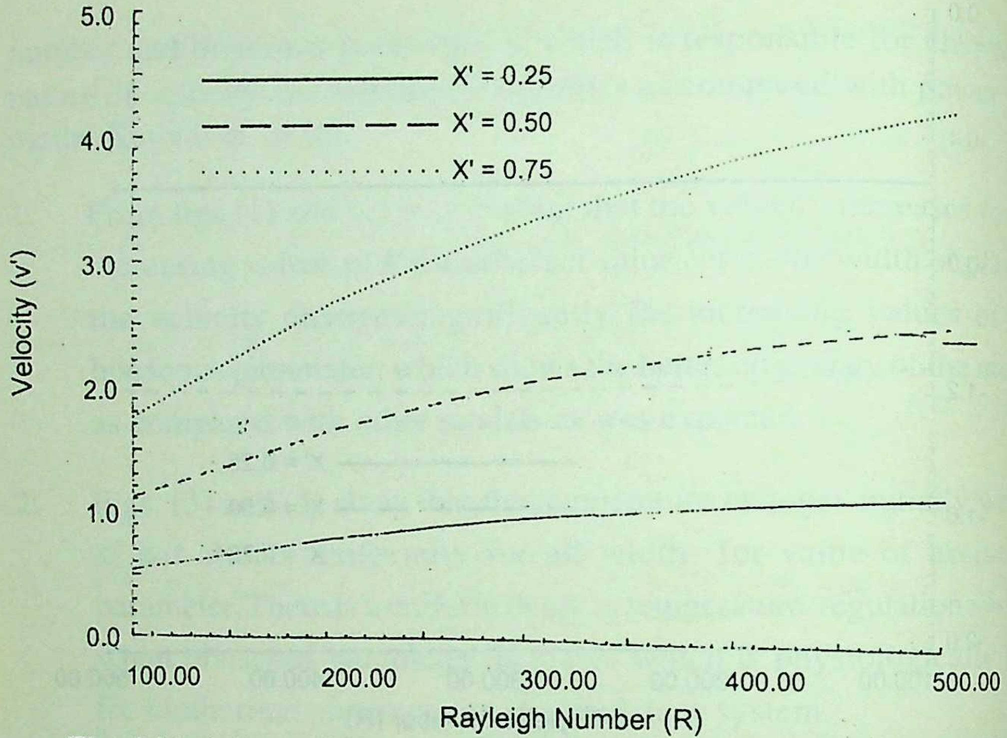

Fig. 3 Variation of Velocity with Rayleigh Number (R) for λ = 1.0

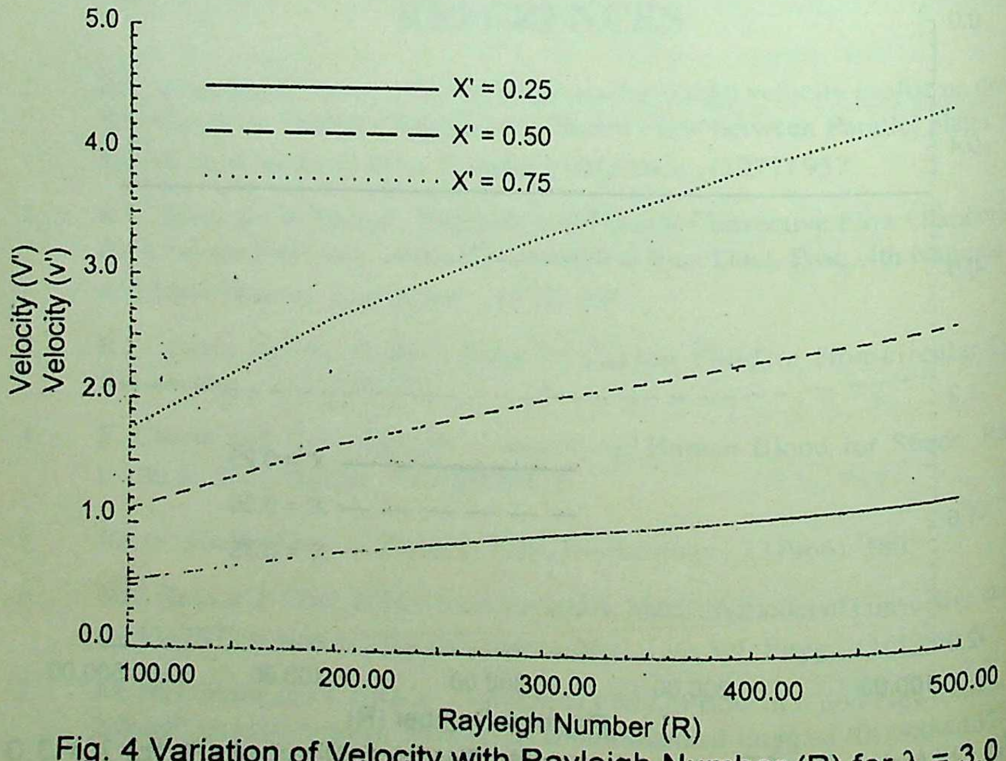

Fig. 4 Variation of Velocity with Rayleigh Number (R) for λ = 3.0

Journal of Natural & Physical Sciences Vol. 11 (1997) 65-78

RECENT STUDIES INTEGRAL TRANSFORMS †

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(Received 1996, Revised 8.8.1997)

A brief survey of work being done by author and band of associated workers is presented. This includes work on Special functions, Integral transforms, Fractional Calculus, extension of Special functions to those with matrix argument and extension of Integral Transforms to generalized functions space and with matrix argument.

The theory of integral transforms in the classical sense can be traced back to about 200 years. But the theory of distributions is only seventy five years old. Some elements of the theory may be found in a paper of Sobolev [50]. He actually tried to generalize the function concept and the classical operations to a larger field in which the Cauchy problem can be solved.

The papers of L. Schwartz [51, 52, 53, 54, 55, 56] gave a firm footing to the theory of distributions. It has been rightly stated by a reputed mathematician that "twentieth century may appropriately be called the century of Functional Analysis". It may be pointed out that the theory of distributions occupies a very important role in the field of application of functional analysis. During last fifty

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† A major part of this work was presented at tenth Annual Conference of the Ramanujan Mathematical Society (Rishikesh 1995).

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years the phenomenal growth in the theory of partial differentia equations is the result of theory of distributions which has acted a a catalytic agent.

Besides Schwartz others such as Siroski [57] and Minkusinski Temple [58], Ravetz (59). Bremermann [60] Kothe [61], Tillman [62] and Sato [63] gave other approaches to distribution theory

It all started with giving mathematically accepted definitio to Dirac's Delta function defined by

$$\delta(x) = 0 \text{ for } x \neq 0$$

and $\int_0^\infty \delta(x)dx = 1$

This function was used to represent mathematically the concep of a shock, like the impact of hammer in mechanics or a larg voltage of very small duration in electrical engineering. But it i not a function in the strict sense. For in the theory of integrals if function is zero almost everywhere in an interval, its integral ove that interval is also zero. Actually Delta function is a functional

As an elementary definition we denote by K the set of all rea functions $\phi(x)$ with continuous derivatives of all orders and wit bounded support (The support of a continuous function $\phi(x)$ i the closure of the set on which $\phi(x) \neq 0$)

We call these functions the test functions and call K the spac of test functions. We then define a generalized function as an linear continuous functional defined on K. The terms 'distribution and generalized functions are used by most of the mathematician as synonymous but *A.H.* Zomanian makes a distinction betwee these two terms. The term 'distribution' is reserved for members o a subspace of a space of generalized functions f whose elemen can be identified with f in a one-one fashion. Thus every distribution is a generalized function but not conversely [64].

An interesting property of distributions is that each distribution is differentiable & derivative, in general, is not a function but a distribution. Also where as in classical analysis continuous functions are not necessarily differentiable but in space of distributions each continuous function becomes differentiable. Its derivative is a distribution. For instance the non differentiable Wierstrass function is differentiable in the sense of distributions but its derivative is not a function but a distribution. Every primitive of a distribution is a distribution.

There are mainly three ways in which a classical integral transform, say,

$$f(s) = \int_I K(s,t)\ \phi(t)\ dt$$

can be generalized.

In the first instance kernel of the trnasform can be replaced by a more general Special function. For example the kernel e^{-st} of well-known Laplace transform has been replaced by $(st)_1 F_1(\beta+\eta+1; \alpha+\beta+\eta+1; -st)$ as has been done by me in introducing Dr. S.M. Joshi generalized Laplace transform (named after my revered father Dr. S.M. Joshi, who inspired me for research) [4,5,7,8,26,64]. Though kernel of the above transform was obtained by Erdelyi by the use of operators of fractional integration, as a special case of a more general kernel, but as a transform it was introduced by me by giving its existence and convergence theorems. Abelian theorems, complex inversion theorems and real inversion theorems were developed by me for the same. [1,2,3].

Similarly the widely known Stieltjes transform

$$f(s) = \int_0^\infty \frac{dt}{s+t} f(t)dt$$

was generalized by me with the nomenclature Dr. S.M. Joshi generalized Stieltjes transform [1,4,6,7] in the form

$$S^{ac} b(f) = \frac{\Gamma(a)\Gamma(c)}{\Gamma(b)\ s} \int_0^\infty (t/s)^{c-1}\ 2^F 1\ \left(a,c;b;-\frac{t}{s}\right) f(t)\ dt.$$

In this notation Dr. S.M. Joshi generalized Laplace transform can be written $S_b^a(f)$

Dr. S.M. Joshi generalized Stieltjes transform is well established in literature [6, 65, 66 (p. 132)]. Similarly Dr. S.M. Joshi Generalized Hankel transform

$$Sf = \int_0^\infty e^{ist}\ {}_1F_1\ (a;\ b;\ -2ist)\ f(t)dt$$

was introduced by us in 1989 [33].
Also Beta transform was introduced in [68] by us.

It may be pointed out that Dr. S.M. Joshi generalized transform was first discussed by the author in [1958] and its convergence theorems [7], Abelian the theorems [4,7], Real inversion theorems [2], Complex inversion theorem [3] were developed besides introducing Dr. S.M. Joshi generalized Laplace transform of two variables [4,7]. Dr. S.M. Joshi generalized transform as a convolution transform, Fractional integration and Dr. S.M. Joshi generalized Laplace transform [5] and Fractional integration and generalized Hankel transform [1, 67] were discussed by me.

Dr. S.M. Joshi generalized transform was extended to distributions in a slightly changed form in [26].

It may be pointed out that there are mainly three ways in which a classical integral transform may be extended to generalized functions. In the first method the kernel containing testing function space say S(I), is constructed, imposing as many restrictions on it as necessary. In the second way kernel is to be a function of real variable, say x istead of s. We then construct a testing function space $S(I)$ not necessarily containing the kernel, which is such that $\phi(x) \rightarrow \overline{\phi(x)}$ is an isomerphism from $S(I)$ to another testing function space $\overline{S(I)}$, when there exists a Parsewal relation of the type

$$\int_I [K(\psi)]\,(t)\,\phi\,(t)dt = \int_I \psi(x)\,\overline{\phi\,(x)}dx$$

with suitable restrictions on ϕ and ψ Then $\overline{S}'(\mathrm{I})$ is the required space of distributions and transform of generalized functions in defined by

$$< Kf, \phi > = < f, \phi >$$

There is yet another method of generalizing a transform with certain kernel K_2 by the help of another transform with kernol K_1 which can be generalized by first method. A form of generalized Stieltjes transform introduced by me was discussed and extended to distributions in [11] and in [14] .

In [33] Dr. S.M. Joshi generalized Hankel transform was extended to distributions. In the same work explicit error terms for asymptotic expansions of Dr. S.M. Joshi generalized Stieltjes transform and Dr. S.M. Joshi generalized Laplace transform were discussed. The researcher also found Abelian theorems, Complex inversion theorems etc. for Dr. S.M. Joshi generalized Hankel transform. Haq, I. discussed the transform

$$f(s) = \int_0^\infty e^{-st} (st)^{C+1} \, \psi\,(a-b;\, 2+c-b;\, st)\, \phi(t)dt$$

in generalized function space [23, 32] Nayal [45-48, 67] discussed Laplace-Hardy transform in detail. Upreti, *R.* discussed a moment problems, distributional Beta transform, Gamma type operators [17,28, 68].

As early as 1952 Bochner introduced Bessel function of matrix argument in connection with a lattice point problem [69]. Herz [70] extended his work and generalized the classical special functions of hypergeemetric type to functions of matrix agruments. He gave the integral representation of the hypergeometric function with the help of generalized Laplace and its inverse. One of the more exciting results in the multivariate analysis has been the discovery of zonal polynonials by James [71, 72] Constantine [73] showed that zonal polynonials form a basis of a meaningful generalizations of whole family of hypergeometric functions to matrix argument. Besides others Mathai & Saksena [76] have studied the subject in detail R.M. Joshi discussed [27] the theory of special functions and integral transforms with matrix arguments. In particular we discussed inversion theorem for Laplace transform and extension of Confluent Hypergeometric function of Second kind [31]. We also extended Dr. S.M. Joshi Generalized transform to matrix argument [74] Some other integral representations have also been obtained. Operational calculus for Laplace transform of matrix agrument has been developed by us. Dhami [9] has also studied matrix argument analysis in his thesis. Some theorems for special functions have been discussed by him [15, 16].

Hahn [75] defined basic analogues of well known Laplace transform in 1949. When $q \rightarrow 1$ these reduce to classical Laplace transform Let q be a parametre which in general, is restricted to the condition $|q| < 1$. Basic analogue or q-analogue of a number 'a' is defined by

$$[a] = \frac{1-q^a}{1-q}$$

When $q \to 1$, $[a] \to a$

Similarly q-difference operator and q-integral operators are defined. For definitions etc. see [18]. We have discussed extension of distributional q-analogue of fractional integration operators. Similarly Distributional hasic Laplace and Stiltjes transforms have been developed.

More recently Bisht [77] has studied theory of numerical inversion of integral transforms (including Dr. S.M. Joshi Generalized transforms) discussed above in order to make the same computer oriented as demanded by modern technology. A chapter devoted to evaluating real zeros of the Confluent Hypergeometric function by developing an algorithm and a program in C language. The program runs well under the DOS and Xenix (Unix) environment.

We have tried to study statistically the style problem in literature [13, 21]. Also Dr. S.M. Joshi generalized Laplace transform has been used in Statistics [34, 36, 38]. It has been extended to Lerentz spaces [22, 35]. Q-analogues of P.J. operators (defined in memory of my Respected Mother Smt. Parwavi Joshi) have been developed and discussed in [37].

Other activities under the guidance of the author include research in fixed point theorems [42, 44] by R.P. Pant and on extreme Forms by C.S. Bisht [43]. An important paper (presented at International Conference in Tokyo) [39, 40] indicated the use of fractional calculus to develop a new formula (called S.M.P.J. formula in memory of my Respected Father and Respected Mother) for sum of powers of natural numbers. This has been recently appriciated by Prof. S.C. Woon of Deptt. of Applied Maths & Physics, Cambridge University in his letter to me.

A bibliography of work being done at this centre also appears in [41] which was presented at International conference at Koriyama (Japan).

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Journal of Natural & Physical Sciences Vol. 11 (1997) 79-86

AN ANALYSIS OF WATER BEING SUPPLIED TO GURUKULA KANGRI CAMPUS, HARDWAR

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(Received 29.5.1997; Revised 29.9.1997)

ABSTRACT

The water being supplied from different bodies to the various residential colonies, offices, laboratories etc. situated inside the campus of Gurukula Kangri University, Hardwar, has been analysed for a number of Physico-Chemical and Biological parameters and the results have been interpretated and compared with the standards prescribed by USPHS, IS, WHO and ICMR.

Keywords: Water analysis, Gurukula Kangri Hardwar Campus.

INTRODUCTION

Gurukula Kangri University is situated on Hardwar Delhi highway, just about 6 Km away from the famous Har-ki-pouri in Hardwar.

Keeping in view the water pollution problem, the analysis of water being supplied to the houses, offices, laboratories and cattle houses situated inside the campus of. Gurukula Kangri University, was under taken by us with an aim to provide suggestions for the improvement of water quality. There are four main sources of water supplies in the Gurukula Kangri University campus:-

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1. Municipal water supplies to professors' quarters at "Satya Shikshak Sadan" .(constituting the sample being referred to as sample no. 1 in the following pages.)

2. Water being supplied from the tube well adjacent to the vice-Chancellor's office) (being referred to as sample number 2 in the following pages). This water is supplied to the laboratories, departments and office in faculty of science and Registrar's office.

3. Supplies from the tube well situated in Gurukula (i.e. Children wing of university) to " Chhota parivar colony", "Bada parivar colony" and the children's hostel (being referred to as sample No.3 here after).

4. Supplies from the tube well/tank in" Gaushala". This water is being supplied to Gaushala and few houses in the " Chhota parivar colony" (being referred to as sample number (4) here after).

The water is being utilized by campus residents for domestic purposes, gardening, laboratory work, and for their cattle. However besides it, there is one more source of water supplies in university campus and that is from the Ganga canal, which is being utilized for the irrigation purposes in the agricultural fields of the university. This has not been included in the present studies.

MATERIAL AND METHODS

All chemicals used were of A.R./G.R. grade, "A" certificate pipettes were used. All other glassware used were of borosil glass. Composite and integrated samples were collected from different sources. Polyethylene bottles were used for this purpose . However, glass bottles were also used for the collection of samples for the purpose of dissolved oxygen analysis. The samples were analysed for different physico-chemical parameters as per standard procedure[1-3]. Dissolved oxygen was analysed by winkler's method [6].

"Systronics -324" pH meter, "Systronics conductometer-305", "Systronics nephelo-turbidimeter-131" and "Systronics Flame photometer Mediflame-127" were used for measurement of pH, conductance, turbidity, sodium and potassium contents in the samples respectively.

RESULTS AND DISCUSSION

The analysed parameters for the four samples under consideration, are given in table -1. The domestic water supplies as suggested by USPHS and IS [1,4] are shown in the table -2. The IS values, available only for the few parameters, are much higher than those for USPHS, obviously for no good reasons. So some ICMR and WHO [5] values are also shown in table-2.

A perusal of the data and its comparison with the standard values reveals that the colour, odour, taste and turbidity are within the permissible limits. However, the specific conductance reveals that in the sample no. 4 and 3 the concentration of ions is much higher than permitted but it causes no worry because this water is not being used for agricultural purposes. Total hardness is also within permissible limit according to the WHO standards. The results of DO, COD, ammonia, sulphate, nitrite, nitrate, calcium, inorganic phosphorus, chlorides, total solids, total dissolved solids, chromium and mercury are well within the permissible concentrations.

In all samples the magnesium and suspended solids are in excess to the standard value. In sample no. 1 and 4, iron is in excess to the standard value. However sample No.2 and 3 show the iron contents within permissible limit. However, the suspended solids are not in so much high concentration that the sample can be rejected. Higher values for iron and magnesium concentrations are perhaps due to the fact that the water tanks and the distribution pipe lines are very old besides being rarely cleaned and are also reported to be in damaged conditions at a few points.

Table-1
Results of Sample Analysis

S.No.	Parameters	Sample No.1	Sample No.2	Sample No.3	Sample No.4
1.	Temperature	25^0C	26^0C	25^0C	27^0C
2.	Color	Colour less	Colour less	Colour less	Colour less
3.	Odour	n.d.	n.d	n.d.	n.d.
4.	Turbidity (NTU)	0.5	2.0	1.0	1.5
5.	pH	7.6	7.3	8.0	7.6
6.	Conductivity (μmhos)	1.3x10^2	0.9x10^2	3.2x10^2	4.2x10^2
7.	Dissolved oxygen (mg/L)	6.08	6.40	3.60	4.40
8.	Biochemical oxygen demand(mg/L)	2.84	1.13	1.24	0.35
9.	Chemical oxygen demand (mg/L)	11.2	7.2	9.6	8.0
10.	Hardness (mg/L)	98.0	104	102	128
11.	Calcium (mg/L)	94	82	96	88
12.	Magnesium (mg/L)	40.70	49.95	42.50	52.62
13.	Total dissolved solids(mg/L)	290	240	320	200
14.	Suspended solids (mg/L)	10	10	30	30
15.	Total solids (mg/L)	300	250	350	230
16.	Chloride (mg/L)	24.14	21.30	17.04	17.04

Table-1 Continued

Table-1 Continued

S.No.	Parameters	Sample No.1	Sample No.2	Sample No.3	Sample No.4
17.	Residual Chloride (mg/L)x 10^2	1.775	3.550	1.775	5.325
18.	Oil and Grease (mg/L)	0.0	0.0	0.0	0.0
19.	Sulphate (mg/L)	15.5	16.0	16.5	8.0
20.	Nitrate (mg/L)	0.38	0.27	0.81	0.28
21.	Nitrite(mg/L)	0.051	0.019	0.023	0.051
22.	Inorganic Phosphorus (mg/L)	0.0	0.15	0.11	0.0
23.	Total Phosphorus (mg/L)	0.30	0.32	0.27	0.24
24.	Organic phosphorus(mg/L)	0.30	0.17	0.16	0.24
25.	Ammonia (mg/L)	0.28	0.0	0.0	0.0
26	Total Kjeldahl nitrogen(mg/L)	0.0	0.0	0.0	0.0
27.	Organic nitrogen (mg/L)	0.28	0.0	0.0	0.0
28.	Total alkalinity (mg/L)	235	250	260	245
29.	Acidity (mg/L)	62.5	107.5	72.5	45.0
30	Sodium (mg/L)	53	65	68	41
31.	Potassium (mg/L)	1.3	1.7	1.6	1.6
32.	Chromium (mg/L)	0.050	0.016	0.024	0.040
33.	Iron (mg/L)	0.42	0.23	0.23	0.34
34.	Mercury (mg/L)	0.0	0.0	0.0	0.0
35.	Colliform cells per 100 ml	0.0	6.63	4.53	9.25

n.d. = nothing disagreeable

Table-2
Standards of Water Quality [4,8,9]
(Water For Drinking Purpose)

S.No.	Parameters	USPHS	Bureau of Indian Standards (IS-10500 : 1991)		ICMR	
			Requirement Desirable Limit	Permissible Limit in the absence of alternate source	Highest desirable level	Maximum Permissible level
1.	Colour	Colourless	5	--	25.0 units	25.0 units
2.	Odour	Odourless	Unobjectionable	--	Unobjectionable	Unobjectionable
3.	Taste	Tasteless	Agreeable	10.0	Unobjectionable	Unobjectionable
4.	Turbidity	5.0 NTU	10.0 NTU	10.0 NTU	5.0 JTU	25.0JTU
5.	pH	6.0-8.5	6.5-8.5	No relaxtion	7.0-8.5	6.5-9.2
6.	Dissolved oxygen (mg/L)	4.0-6.0	--	--	--	--
7.	Specific Conductance	300 μmho cm^{-1}	--	--	--	--
8.	Chemical oxygen demand (mg/L)	4.0	--	--	--	--
9.	*Total hardness (mg/L)	--	300.0	600.0	300.0	600.0
10.	Total dissolved solid (mg/L)	500.0	500.0	2000.0	500.0	1500.0**
11.	Suspended solid (mg/L)	5.0	--	--	--	--
12.	Total solid (mg/L)	505.0	--	--	--	--
13.	Chloride (mg/L)	250.0	250.0	1000.0	200.0	1000.0

Table-2 Continued

Table-2 Continued

S.No.	Parameters	USPHS	Bureau of Indian Standards (IS-10500 : 1991) Requirement Desirable Limit	Bureau of Indian Standards (IS-10500 : 1991) Permissible Limit in the absence of alternate source	ICMR Highest desirable level	ICMR Maximum Permissible level
14.	Sulphate (mg/L)	250.0	200.0	400.0	200.0	400.0
15.	Nitrate (mg/L)	45.0	45.0	45.0	20	***
16.	Phosphate (mg/L)	0.1	--	--	--	--
17.	Ammonia (mg/L)	0.5	--	--	--	--
18.	Calcium (mg/L)	100	75.0	200.0	75.0	200.0
19.	Magnesium (mg/L)	30	30.0	100.0	<50 if sulphate 200	--
20.	Chromium (mg/L)	0.05	0.05	No relaxation	--	--
21.	Iron (mg/L)	<0.3	0.3	1.0	0.1	1.0
22.	Mercury (mg/L)	0.001	0.001	No relaxation	--	0.001
23.	Colliform Cells/ 100ml	1	10	--	--	10
24.	Residual Chlorine (mg/L)	--	0.2	--	--	--
25.	Total alkalinity (mg/L)	--	200	600	--	--
26.	Chromium (mg/L)	--	0.05	No relaxation	--	--

(*) WHO [5]; Maximum permissible value = 500 mg/l on $CaCo_3$ scale

(**) [8]; Dissolved solids relaxable upto 3000 mg/l in case where alternate sources are not available within reach.

(***) [8]; more information is required to prescribe a value but in no circumstances should exceed 100 mg No_3/l.

As far as the colliform cells per 100 ml results are concerned, the water being supplied by municipal committee is quite good but the water being supplied from other reservoirs (i.e. sample no. 2, 3 and sample No.4) is expected to contain higher amounts of organic matter, especially the colliform group of microorganisms. So it is not good for the human health and a proper treatment of these water supplies is immediately desirable [7]

It is clear that the water being supplied by municipal committee is comparatively better than the water being supplied from the other reservoirs in the campus of Gurukula Kangri University. Particularly the Gaushala resevoir tank requires thorough cleansing. Further the distribution pipe lines should be repaired or replaced to avoid the leakage of soil or organic matter into the water which is definetely responsible for higher value of colliform cells in these water supplies.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 87-94

ROUND-OFF STABILITY OF ITERATION PROCEDUERS FOR OPERATORS IN b-METRIC SPACES

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(Received 3.12.1997)

ABSTRACT

A.M. Ostrowski's type stability theorems of iteration procedures for Banach operators in generalized metric spaces, so-called *b*-metric spaces, are presented.

Mathematics Subject Classifications (1991) : 47H10, 47H15, 65D15.

Keywords : Stable iteration, Banach contraction, fixed point *of operators*, *b*-metric spaces.

INTRODUCTION

Consider a metric space (X, d) and a mapping $T : X \rightarrow X$. The concept of iteration procedure for finding a fixed point of T is very important and fruitfull in many areas of mathematics. However, in computation, an approximate sequence $\{y_n\}$ is used in place of a sequence of successive approximations. This leads to an idea of a fixed point iteration procedure stable with respect to T. The first stability result in metric spaces has been presented by A.M.

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Ostrowski [4]. His theorem has been extended by many mathematicians to various classes of operators (see e.g.[3], [5]-[8]).

It this paper we discuss the problem of stability of iteration procedure for operators in so called "*b-metric spaces*".

PRELIMINARY RESULTS

Now we introduce a generalization of ordinary metric space (see Czerwik [2]).

Definition 1 : Let X be a set and let $s \geq 1$ be a given real number. A function $d : X \times X \to R_+$ (the set of nonnegative real numbers) is said to be a b-metric iff for all $x, y, z \in X$ the following conditions are fullfield :

$$d(x,y) = 0 \text{ iff } x=y, \tag{1}$$

$$d(x,y) = d(y,x), \tag{2}$$

$$d(x,z) \leq s\,[d(x,y)+d(y,z)]. \tag{3}$$

Then (X,d) is called a b-metric space.

In [1] *S*. Czerwik has proved the following result about the existence and uniquenness of fixed points of operators of Banach type in b-metric spaces.

For $T : X \to X$ we denote by T^n the n-th iterate of T.

Theorem 1 : ([1]) let (X,d) be a complete b-metric space and let $T : X \to X$ satisfy

$$d[Tx,Ty] \leq \varnothing[d(x,y)],\ x,y \in X, \tag{4}$$

where $\varnothing : R_+ \to R_+$ is an increasing function such that $\lim_{n\to\infty} \varnothing^n(t) = 0$ for each fixed $t > 0$. Then T has exactly one fixed point u (i.e. $Tu=u$) and

$$\lim_{n\to\infty} d[T^n x, u] = 0 \text{ for each } x \in X. \tag{5}$$

Remark 1 : Theorem 1 has been proved for $s = 2$, but the proof remains unchanged for any $s \geq 1$.

First we prove the following.

Lemma : Let $\{\varepsilon_n\}$ be a sequence of nonnegative real numbers. Then

$$\lim_{n\to\infty} \varepsilon_n = 0 \quad \text{iff} \quad \lim_{n\to\infty} S_n = 0 \tag{6}$$

where

$$S_n = \sum_{r=0}^{n} \alpha^{n-r} \varepsilon_r \tag{7}$$

and

$$0 \leq \alpha < 1. \tag{8}$$

Proof : First assume that $\lim_{n\to\infty} S_n \to 0$. Then since $0 \leq \varepsilon_n \leq S_n$, we get $\lim_{n\to\infty} \varepsilon_n = 0$. Now, suppose $\varepsilon_n \to 0$ as $n \to \infty$. Take arbitrary $\varepsilon > 0$. We can consider $\alpha > 0$. Put $\beta = \alpha^{-1}$. There exists natural m such that $\varepsilon_n < \varepsilon$ for all $n > m$. Now we get the inequalities (for $n = m+k$, $k \in N$)

$$S_n = \alpha^n[\varepsilon_0 + \beta\varepsilon_1 + \ldots + \beta^n \varepsilon_n] = \alpha^n[(\varepsilon_0 + \ldots + \beta^m \varepsilon_m) + (\beta^{m+1}\varepsilon_{m+1} + \ldots + \beta^{m+k}\varepsilon_{m+k})]$$

$$\leq \alpha^n(\varepsilon_0 + \ldots + \beta^m \varepsilon_m) + \varepsilon\alpha^n(\beta^{m+1} + \ldots + \beta^{m+k}) \leq \alpha^n(\varepsilon_0 + \ldots + \beta^m \varepsilon_m) + \varepsilon\,(1-\alpha)^{-1}.$$

Therefore, there exists $n_0 \geq m$ such that for $n > n_0$ we have

$$S_n \leq \varepsilon + \varepsilon(1-\alpha)^{-1}.$$

Since $S_n \geq 0$ and $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty} S_n = 0$.

This completes the proof.

STABILITY OF OPERATORS

Let X be a b-metric space and $T : X \to X$ an operator. For any $x_0 \in X$, we define (Picard iterative procedure)

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots \tag{9}$$

Let the sequence $\{x_n\}$ be convergent to a fixed point u of T. Let $\{y_n\}$ be an arbitrary sequence of elements of X and set $\varepsilon_n = d\,(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \ldots$.

The iteration process (9) is said to be T-stable or stable with respect to T (cf. [3]) provided that $\lim\limits_{n\to\infty} \varepsilon_n = 0$ implies that $\lim\limits_{n\to\infty} y_n = 0$.

Now we present the following result.

Theorem 2 : Let X be a complete b-metric space and $T : X \to X$ satisfy

$$d[Tx,Ty] \le gd\,(x,y) \tag{10}$$

for all $x,y \in X$ where

$$\alpha := sg < 1. \tag{11}$$

Let x_0 be an arbitrary point in X and $\{x_n\}$ a sequence of iterates of T given by (9). Let $\{y_n\}_{n=0}^{\infty}$ be a sequence in X, and set $\varepsilon_n = d[y_{n+1}, Ty_n]$, $n = 0, 1, 2, \ldots$.

Then

$$\lim_{n\to\infty} x_n = u = Tu, \tag{12}$$

$$d(u,y_{n+1}) \le sd(u,x_{n+1}) + s\alpha^{n+1} d(x_0,y_0) + s^2 \sum_{r=0}^{n} \alpha^{n-r} \varepsilon_r. \tag{13}$$

Moreover,

$$\lim_{n\to\infty} y_n = u \quad \text{iff} \quad \lim_{n\to\infty} \varepsilon_n = 0. \tag{14}$$

Proof : From Theorem 1 for $\phi(t) = gt$ we get that T has exactly one fixed point $u \in X$, i.e. $Tu = u$ and the sequence $\{x_n\}$ has the limit u, which means that (12) holds true.

Now, in view of (10) and the traingle inequality (3) for b-metric, we obtain for nonnegative integer n,

$$d(x_{n+1},y_{n+1}) = d\ (Tx_n,\ y_{n+1}) \le sd(Tx_n,\ Ty_n) + sd(Ty_n,\ y_{n+1}) \le sgd(x_n,\ y_n) + s\varepsilon_n$$

$$\le sg[sgd\ (x_{n-1},\ y_{n-1}) + s\ \varepsilon_{n-1}] + s\ \varepsilon_n \le \alpha^2 d(x_{n-1},\ y_{n-1}) + s[\alpha_{n-1} + \varepsilon_n].$$

Consequently, by induction principle, we get the inequality

$$d(x_{n+1},\ y_{n+1}) \le \alpha^{n+1} d(x_0,\ y_0) + s\sum_{r=0}^{n} \alpha^{n-r}\varepsilon_r.$$

Therefore, we have

$$d(u,\ y_{n+1}) \le sd(u,\ x_{n+1}) + sd(x_{n+1},\ y_{n+1}) \le sd(u,x_{n+1}) + s\alpha^{n+1} d\ (x_0,y_0) + s^2\sum_{r=0}^{n} \alpha^{n-r}\varepsilon_r$$

i.e. the estimation relation (13).

To prove (14), assume that $y_n \to u$ as $n \to \infty$. Then we obtain $\varepsilon_n = d(y_{n+1},\ Ty_n) \le sd(y_{n+1},\ u) + sd(u,\ Ty_n) \le sd(y_{n+1},\ u) + s[\ sd(u,\ Tu) + sd(\ Tu,\ Ty_n)] \le sd(y_{n+1},\ u) + s^2gd(u,\ y_n)$.

Therefore the assumption $y_n \to u$ as $n \to \infty$ implies $\varepsilon_n \to 0$ as $n \to \infty$.

Conversely, let $\varepsilon_n \to 0$ as $n \to \infty$. Since $x_n \to u$ as $n \to \infty$ and $0 \le \alpha = sg < 1$, we see that the first two terms of the right hand side of (13) vanish in the limit.

In view of (6) of Lemma the third term of (13) has the limit zero. This completes the proof.

In the next theorem instead of condition (11), we consider an appropiate iterate of the operator T.

Theorem 3 : Let X be a complete b-metric space and let $T{:}X \to X$ satisfy.

$$d(Tx, Ty) \le g d(x, y),\ x, y \in X \text{ and } 0 \le g < 1. \tag{15}$$

Let m be a natural number such that

$$\beta : g^m s < 1 \tag{16}$$

and set $F := T^m$.

For $x_0 X$, let $x_{n+1} = Fx_n$, $n = 0, 1, 2, \ldots$ let $\{y_n\}_{n=0}^{\infty}$ be a sequence in X and $\varepsilon^*_n = d(y_{n+1}, Fy_n)$, $n = 0, 1, 2, \ldots$. Then T and F have exactly one common fixed point u and

$u = \lim_{n\to\infty} x_n$.

Moreover,

$$d(u, y_{n+1}) \le sd(u, x_{n+1}) + s\beta^{n+1} d(x_0, y_0) + s^2 \sum_{r=0}^{n} \beta^{n-r} \varepsilon^*_r, \tag{17}$$

$$\lim_{n\to\infty} y_n = u \quad \text{if} \quad \lim_{n\to\infty} \varepsilon^*_n = 0. \tag{18}$$

Proof : Take $\phi(t) = gt$, $t > 0$, then from Theorem 1 it follows that T has exactly one fixed point $u \in X$. Now, we have for $x, y \in X$,

$$d[T^m x, T^m y] \leq gd[T^{m-1}x, T^{m-1}y] \leq g^2 d[T^{m-2}x, T^{m-2}y] \leq \ldots \leq g^m d(x,y).$$

This means that

$d(Fx,Fy) \leq g^m d(x,y)$ for all $x, u \in X$. Again, from Theorem 1 for $\phi(t) = g^m t$, $t > 0$, we see that F has exactly one fixed point $w \in X$ and $w = \lim_{n\to\infty} x_n$. We show that $u = w$. Indeed, $u = Tu = T^2 u = \ldots = T^r u = Fu$. T and F has only one fixed point $u = w$.

The statements (17) and (18) are easily verified as in the proof of Theorem 2.

Remark 2 : Under the assumptions of Theorem 3, $\lim_{n\to\infty} \varepsilon^*_n = 0$ implies

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0.$$

Indeed, since $Tu = u$, we have

$$\varepsilon_n = d(y_{n+1}, Ty_n) \leq sd(y_{n+1}, Tu) + sd(Tu, Ty_n) \leq sd(y_{n+1}, u) + sgd(u, y_n) \to 0$$

as $n \to \infty$.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 95-106

PROPAGATION OF CORRELATIONS IN DECAY PROCESS OF HYDRODYNAMIC AND MHD TURBULENCE BEFORE THE FINAL PERIOD

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(Received 8.12.1997)

ABSTRACT

The present paper reports a theoretical calculation of correlation functions. The decay of MHD turbulence at times before the final period is discussed.

Keywords : MHD turbulence, correlation functions. Decay process.

INTRODUCTION

The manner in which a fluid in turbulent motion transports and modifies the initial shape of physical quantities is of considerable importance in many fields. If we agree to call the motion of physical quantities in turbulent fluids dispersion, we may cite many example of this process. the motion of clouds in the sky, smoke from a smokestack, salt concentration in the sea, electron density in the ionosphere, staruparticles in galactic clouds, and temperature in interstellar regions are but a few example of the turbulent dispersion phenomenon. In these years many authors studies the temperative dispersion.

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T. Dixit [2] consider the effect of corriolis force on the thermal Decay process of M H D turbulence. Sarkar and Kishore [1] studied the decay of M H D turbulence at times before the final period. In this paper we have discussed the decay of M H D dusty turbulence before the final period.

Two Point Correlaion and special Eqations

Let u_i denote the component of velocity at point $p(x_i)$ then

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k}(u_i u_k - h_i h_k) = -\frac{\partial w}{a x_i} + \nu \nu^2_x . u_j + f(v_i - u_i) \quad (3.1)$$

where $w = P/\rho + \frac{1}{2} \; h^2$ is total MHD pressure, $p(x,t)$ is the hydromagnetic pressure.

The corresponding equation for the point p' will be

$$\frac{\partial u'_j}{\partial t} + \frac{\partial}{\partial x'_k}(u'_i u'_k - h'_i h'_k) = \frac{\partial w'}{\partial x'_j} + \nu \nu^2_{x'} . u'_j + f(v'_j - u'_j) \quad (3.2)$$

Multiplying (3.1) by u'_j and (3.2) by u_i, adding and taking ensemble average, we get

$$\frac{\partial}{\partial t}\overline{u_i u_j} + \frac{\partial}{\partial x_k}\overline{(u_i u_k u'_j - h_i h_k u'_j)} + \frac{\partial}{\partial x'_k}(u'_i u'_k u_i - \overline{h'_i h'_k u_i})$$

$$= -\frac{\partial}{\partial x'_j}\overline{w' u_i} - \frac{\partial}{\partial x'_i}\overline{w u'_j} + \nu \nu^2_x \overline{u'_j u_i} + \nu \nu^2_x \overline{u_i u'_j} + f(u_i u'_j - u_i u'_j$$

$$+ \overline{u'_j u_i} - \overline{u_i u'_j} \qquad \ldots(3.3)$$

using the transformation

$$\frac{\partial}{\partial r_k} = - \frac{\partial}{\partial x_k} = \frac{\partial}{\partial x'_k}$$

$$r_i = x'_i - x_i, \quad \nu^2_{x'} = \nu^2$$

and relations [3]

$$\overline{u_k h_i h'_j} = - \overline{h_i u'_k h'_j}, \quad \overline{u_i u_j u'_k} = - \overline{u'_i u'_j u_k}$$

and

$$\overline{h_i h'_j h'_k} = - \overline{u_i h_k h'_j}, \quad \overline{h_i h_j u'_k} = - \overline{h'_i h'_j u_k}$$

We consider that dust particles are non conducting and hence $\overline{h_i v'_j}$ $= h'_j u'_i = 0$. We also assume that the instantaneous velocities at the point remain unaffected by the dust particles of the other point, i.e.

$$\overline{u_i v'_j} = \overline{u'_j v_i} = 0.$$

Also $\overline{w' u_i}$ and $\overline{w' v'_j}$ vanish (see 37)

Hence equation (3.3) can be written as

$$\frac{\partial}{\partial t} \overline{u_i u'_j} - 2 \frac{\partial}{\partial r_k} (\overline{u_i u_k u'_j} - \overline{h_i h_k h'_j})$$

$$= 2 \nu \nu^2 \overline{u_i u'_j} - 2f \overline{u_i u'_j} \qquad ..(3.4)$$

Induction equation of a magnetic field at point p is

$$\frac{\partial h_i}{\partial t} + u_k \frac{\partial h_i}{\partial x_k} - h_k \frac{\partial h_i}{\partial x_k} = \frac{\nu}{P_m} \frac{\partial^2 h'_j}{\partial x_k \, \partial x_k} \qquad ..(3.5)$$

where $P_M = \frac{\nu}{\lambda}$ is magnetic Prandtl number, is the magnetic diffusivity, $h_i(x,t)$ the magnetic field fluctuation, uk(x,t) is the turbulent velocity, ν is kinematic viscosity.

The corresponding equation for the point p' will be

$$\frac{\partial h_j}{\partial t} + u'_k \frac{\partial h'_j}{\partial x_k} - h'_i \frac{\partial u'_j}{\partial x'_k} = \frac{\nu}{P_m} \frac{\partial^2 h'_j}{\partial x'_k \, \partial x'_k} \qquad ..(3.6)$$

Multiplying equation (3.5) by h'_j and (3.6) by h_i adding and taking ensemble average, we get

$$\frac{\partial}{\partial t} \overline{h_i h'_j} + \frac{\partial}{\partial x_k} \overline{u_k h_i h'_j} + \frac{\partial}{\partial x'_k} \overline{u'_k h_i h'_j}$$

$$- \frac{\partial}{\partial x_k} \overline{u_i h_k h'_j} - \frac{\partial}{\partial x_k} (\overline{u'_j h_i h_k})$$

$$= \frac{\nu}{P_M} \left(\frac{\partial^2 \overline{h_i h'_j}}{\partial x_k \, \partial x_k} + \frac{\partial^2 \overline{h_i h'_k}}{\partial x_k \, \partial x'_k} \right)$$

or
$$\frac{\partial}{\partial t} \overline{h_i j'_j} + 2 \frac{\partial}{\partial r_k} u'_k h_i h'_j - \frac{\partial}{\partial r_k} \overline{u_i u_k h_j}$$

$$= 2 \frac{\nu}{P_M} \frac{\partial^2}{\partial r_k \, \partial r_k} + \overline{h_i j''_j} \qquad ..(3.7)$$

Let

$$\overline{h_i h'_j (r)} = \int_{-\infty}^{\infty} \Psi_i \Psi'_j (k) \exp (ik.r) \, dk.$$

$$\overline{u_k h_k h'_k (r)} = \int_{-\infty}^{\infty} \alpha_i \Psi_k \Psi'_j (k) \exp (ik.r) \, dk.$$

Interchanging the subscripts i and j and then interchange the points P and P' we have

$$\begin{aligned}\overline{u'_k h_i h'_j(r)} &= \overline{u_k h_i h'_j(-r)} \\ &= \int_{-\infty}^{\infty} \alpha_k \psi_i \psi'_j(-k) \ \exp\ (ik.r)\, dk.\end{aligned}$$

Therefore (3.7) can be written as

$$\frac{\partial}{\partial t} \overline{\psi_i \psi'_j(k)} + 2 \frac{\nu}{P_M} k^2 \overline{\psi_i \psi'_j(k)} = 2i\ k_k \left[\overline{\alpha_i \psi_k \psi_j(k)} - \overline{\alpha_k \psi_i \psi'_j(-k)}\right] \qquad \ldots(3.8)$$

The tensor equation (3.8) becomes a scalar equation by contraction of indicies i and j

$$\frac{\partial}{\partial t} \overline{\psi \psi'_j(k)} + 2 \frac{\nu}{P_M} k^2 \overline{\psi_i \psi'_j(k)} = 2i\ k_k \left[\overline{\alpha_i \psi_k \psi'_j(k)} - \overline{\alpha_k \psi_i \psi'_j(-k)}\right] \qquad \ldots(3.9)$$

THREE POINT CORRELATION AND EQUATIONS

We take momentum equation at the point P and the induction equations of magnetic field fluctuations at p' and p'

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_1}{\partial x_k} - h_k \frac{\partial h_1}{\partial x_k} = - \frac{\partial w}{\partial x_1} + \nu \frac{\partial^2 u_1}{\partial x_k \partial x_k} + f(v_1 - u_1) \qquad \ldots(4.1)$$

$$\frac{\partial h'_i}{\partial t} + u'_k \frac{\partial h'_i}{\partial x'_k} = - h'_k \frac{\partial u'_i}{\partial x'_k} = \frac{\nu}{P_M} \frac{\partial^2 h'_i}{\partial x'_k \partial x'_k} \qquad \ldots(4.2)$$

$$\text{and} \quad \frac{}{\partial t} + u'_k{}' \frac{}{\partial x'_k{}'} - h'_k{}' \frac{}{\partial x'_k{}'} = \frac{}{P_M} \quad \frac{}{\partial x'_k{}' \, \partial x'_k{}'} \qquad \ldots(4.3)$$

Multiplying (4.1) by $h'_i \; h'_j{}'$ (4.2) by $u_1 \; h'_j{}'$ and (4.3) by $u_1 \; h'_i$ adding the three equations and taking averages, we obtain

$$\frac{\partial}{\partial t} \overline{u_1 h'_i h''_j} + \frac{\partial}{\partial x_k} \overline{u_1 u_k h'_j \; h'_j{}'} - \frac{\partial}{\partial x_k} \overline{h_1 h_k h'_r \; h'_j{}'}$$

$$+ \frac{\partial}{\partial x'_k} \overline{u_1 u'_k h'_i \; h'_j{}'} - \frac{\partial}{\partial x'_k} \overline{u_1 u'_i h'_k \; h'_j{}'} + \frac{\partial}{\partial x'_k{}'} \overline{u_1 u'_k{}' h'_i \; h'_j{}'}$$

$$- \frac{\partial}{\partial x'_k{}'} \overline{u_1 u'_j{}' h'_i \; h'_k{}'}$$

$$= - \frac{\partial}{\partial x'_1} \overline{u_i h'_i h'_j{}'} + \frac{\partial^2 u_1 h'_i h'_j{}'}{\partial x_k \; \partial x_k} + \frac{\nu}{P_M} \left[\frac{\partial^2 \overline{u_1 h'_i h'_j{}'}}{\partial x'_k \; \partial x'_k} \right.$$

$$\left. + \frac{\partial^2 \overline{u_1 h'_i h'_j{}'}}{\partial x'_k{}' \; \partial x'_k{}'} \right] + f \; \overline{(\nu_i h'_i h'_j{}'} - \overline{u_1 h'_i h'_j{}'}) \qquad \ldots(4.4)$$

Using the transformations

$$\frac{\partial}{\partial x'_k} = \frac{\partial}{\partial r_k} , \quad \frac{\partial}{\partial x'_k{}'} = \frac{\partial}{\partial r'_k}$$

$$\text{and} \quad \frac{\partial}{\partial x_k} = -\left(\frac{\partial}{\partial r_k} + \frac{\partial}{\partial r'_k}\right)$$

into equation (4.4), we get

$$\frac{\partial}{\partial t} \overline{u_1 h'_i h'_j{}'} - \frac{\nu}{P_M} \left[(1+P_M) \frac{\partial^2 \overline{u_1 h_i h'_j{}'}}{\partial r_k \; \partial r_k} (1+P_M) \frac{\partial^2 \overline{u_1 h'_i h'_j{}'}}{\partial r'_k \; \partial r'_k} \right.$$

$$+ 2P_M \quad \frac{\partial^2 \overline{u_1\, h_i\, h'_j{}'}}{\partial r_k \quad \partial r'_k} = \frac{\partial}{\partial r_k} \overline{u_1\, u_k h'_i h'_j{}'} + \frac{\partial}{\partial r'_k} \overline{u_1\, u_k\, h'_i\, h'_j{}'}$$

$$- \frac{\partial}{\partial r_k} \overline{h_1\, h_k\, h'_i\, h'_j{}'} - \frac{\partial}{\partial r'_k} \overline{h_1\, h_k\, h'_i\, h'_j{}'}$$

$$- \frac{\partial}{\partial r_k} \overline{u_1\, u'_k\, h'_i\, h'_j{}'} + \frac{\partial}{\partial r'_k} \overline{u_1\, h'_k\, h'_j{}'}$$

$$- \frac{\partial}{\partial k'} \overline{u_1\, u'_k{}'h'_i\, h'_j{}'} + \frac{\partial}{\partial r'_k} \overline{u_1 u'_j{}'h'_i\, h'_j{}'} + \frac{\partial}{\partial r'_1} \overline{wh'_i\, h'_j{}'}$$

$$+ \frac{\partial}{\partial r'_i} \overline{wh'_i\ h'_j{}'} \quad + \mathrm{f}\,(\overline{u_1\, h'_i h'_j{}'} - \overline{u_1\, h'_i h'_j{}'})\ f\ (\overline{u_1\, h'_i h'_j{}'} - \overline{u_1\, h'_i h'_j{}'}) \quad \text{...(4.4)}$$

$$\text{Let } \overline{u_1\, h'_i(r)\ h'_j{}'\,(r')} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\phi_1\, \overline{\beta_i(k)\ \beta'_j{}'\,(k')} \exp\, [\overline{i\,(k.r+k'.r')}]\ dkdk'$$

$$\overline{u_1\, u'_k(r) h_i(r)\ h'_j{}'\,(r')} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1 \quad \overline{\phi'_k(k)\ \beta'_j{}'\,(k')\ \beta'_j{}'\,(k')}$$

$$\exp\ [i\,(\overline{k.r+k'.r'})]\ dkdk'$$

$$\overline{u_1\, u'_i(r)\ h'_k(r) h'_j{}'\,(r')} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\phi_i(k)\ \beta_k{}'(k)\ \beta'_j{}'\,(k)\ \beta'_j{}'\,(k')}$$

$$\exp\ [i\,(k.r+k'.r')],\ dkdk'$$

$$\overline{u_1\, u_k h'_i(r)\ h''(r'_j)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1\, \overline{\phi_k \beta_i{}'\ \beta'_j{}'\,(k')}$$

$$\exp\ [i\,(k.r+k,r')],\ dkdk'$$

$$\overline{h_1\, h_k h'_i(r)\ h'_j{}'(r')} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\beta_1\, \beta_k \beta'_i\,(k)\ \beta'_j{}'\,(k')}$$

$$\exp\, [i(k.r + k'.r')]\ dk\ dk'$$

$$\overline{wh'_i(r)\,h'^{\,\prime}_j(r')} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\gamma\beta_i\,(k)\,\beta'^{\,\prime}_j\,(k')}$$

$$\exp\,[i(k.r + k'.r')]\,dk\,dk'$$

$$\overline{\nu_1\,h'_i(r)\,h'^{\,\prime}_j(r')} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\delta_1\,\beta'_i\,(k)\,\beta'^{\,\prime}_j\,(k')}$$

$$\exp.\,[i(k.r + k'.r')]\,dk\,dk'$$

Interchange of points p' and p'' along with the subscripts i and j, results in the relations

$$\overline{u_1\,u'^{\,\prime}_k\,h'^{\,\prime}_j\,h'_i} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{u_1\,u'_k\,h'_i\,h'^{\,\prime}_j}$$

and

$$\overline{u_1\,u'^{\,\prime}_j\,h'_i\,h'^{\,\prime}_k}\,. = \overline{u_1\,u'_i\,h'_k\,h'^{\,\prime}_j}$$

By use of the above facts and relations. The equation (4.5) may b written as

$$\frac{\partial}{\partial t}\overline{\phi_1\,\beta'_i\,\beta'^{\,\prime}_j} + \frac{\nu}{P_M}[(1+P_M)k^2 + (1+P_M)\,k^2\,2P_M\,k_k\,k'_k] + f\,\frac{P_M}{\nu}\,\phi_1\beta'_i\beta'_i$$

$$= i(k_k + k'_k)\;\overline{\phi_1\,\phi_k\beta'_i\,\beta'^{\,\prime}_j} \;-\; i(k_k + k'_k)\;\overline{\beta_1\,\beta_k\,\beta'_i\,\beta'^{\,\prime}_j}$$

$$= i(k_k+k'_k)\;\overline{\phi_1\,\phi_k\beta'_i\,\beta'^{\,\prime}_j} + i(k_k+k'_k)\;\overline{\phi_1\,\phi'_i\,\beta'_k\,\beta'^{\,\prime}_j}$$

$$+\, i(k_1+k'_1)\;\gamma\overline{\beta'_i\,\beta'^{\,\prime}_j} + f\,(\overline{\delta_1\,\beta'_i\,\beta'^{\,\prime}_j}) \qquad \ldots.(4.6$$

The tensor equation (4.6) can be converted to scalar equation by contraction of indices i and j given by

$$\frac{\partial}{\partial t}\,\overline{\phi_1\,\beta'_i\,\beta'^{\,\prime}_j} + \frac{\nu}{P_M}\,[(1+P_M)k^2+k'^2) + 2P_M kk' + f\,\frac{P_M}{\nu}]\;\overline{\phi_1\beta'_i\beta_j}$$

$$= i(k_k + k'_k)\ \overline{\phi_1\,\phi_k\,\beta'_i\ \beta'_j{}'} \ - \ i(k_k + k'_k)\ \overline{\beta_1\,\beta_k\,\beta_1\ \beta'_j{}'}$$

$$- \ i(k_k + k'_k)\ \overline{\phi_1\,\phi'_k\,\beta'_i\ \beta'_L{}'} \ + \ i(k_k + k'_k)\ \overline{\phi_1\,\phi'_i\ \beta_k\,\beta'_i{}'}$$

$$+ \ \gamma\,(k_1 + k'_1)\ \overline{\beta'_i\ \beta'_i{}'} \ + \ f\,\overline{(\delta_1\ \beta'_i\ \beta'_j{}')} \qquad ....(4.7)$$

If we take the derivative with respect to x_1 of the momentum equation (4.1) we have

$$\frac{\partial^2 w}{\partial x_1\,\partial x_1} = \frac{\partial^2}{\partial x_1\,\partial x_k}\ (u_i\,u_k - h_1\ h_k) \qquad ....(4.8)$$

Multiplying equation (4.8) by $h'_i\ h'_j{}'$ taking average and writing equation in terms of the independent variables r and r'

$$- \ \left[\frac{\partial^2}{\partial r_1\ \partial r_1} + \frac{\partial^2}{\partial r'_1\ \partial r'_1} + \frac{\partial^2}{\partial r_1\ \partial r'_1}\right]\ \overline{w\,h'_i\ h'_j{}'}$$

$$= \ \left[\frac{\partial^2}{\partial r_1\ \partial r_k} + \frac{\partial^2}{\partial r'_1\ \partial r'_k} + \frac{\partial^2}{\partial r_1\ \partial r'_k} + \frac{\partial^2}{\partial r'_1\ \partial r'_k}\right]$$

$$\cdot\ [\overline{u_1\ \ u_k\ \ h'_i\ \ h'_j{}'} \ - \ \overline{h_1\,h_k\ \ h'_j{}'}] \qquad ....(4.9)$$

Taking the fourier transforms of equation (4.9) we get

$$- \ r\,\overline{\beta'_i\ \beta'_j{}'} = \frac{(k_1\,k'_k + k'_1\,k_k + k'_{k'} + k'_1\ k'_k)\ (\phi_1\,\phi_k\,\beta'_i\ \beta'_J{}'\,\beta_1\,\beta_k\,\beta'_i\ \beta'_J{}')}{k^2 + k'^2 + 2k_1\,k'_l} \qquad ....(4.10)$$

Equation (4.10) can be used to eliminate $r\,\beta'_i\ \beta'_J{}'$ from equation (4.7).

SOLUTION FOR TIMES BEFORE THE FINAL PERIOD

It is known that the equation for final period decay is obtained by considering the two point correlation after neglecting the third order correlations. To study the decay for times before the final period, the three point correlation equations are considered and the quadruple corelations are neglected. By making use of (4.10) and neglecting quadruple correlations the equation (4.7) can be written as

$$\frac{\partial}{\partial t} \; d_1 \beta'_i {\beta'}_i' + \frac{\nu}{P_M} [(1+P_M) \;\; (k^2+k'^2) + f\frac{P_M}{\nu} + 2P_M kk']$$

$$\phi_1 \beta'_i {\beta'}_i' = 0 \qquad \text{....(5.1)}$$

Intigrating the equation (5.1) between t_o and t with inner multiplication by k_k gives

$$k_k \, \phi_1 \beta'_i {\beta'}_i' = k_k \; \overline{[\phi_1 \beta'_i {\beta'}_1']}_o$$

$$.\exp\left[-\frac{\nu}{P_M} (1+P_M)(k^2+k'^2) + f \frac{P_M}{\nu} + 2P_M k'_k \cos \Theta)(1+t_o)\right] \qquad \text{....(5.2)}$$

where Θ is the angle between k and k' .

Now by letting $r' = 0$ in fourier transforms we have

$$\overline{\alpha_i \Psi_k \Psi'_j(k)} = \int_{-\infty}^{\infty} \overline{\phi_i \beta'_i {\beta'}_i'} \, dk.$$

and

$$\alpha_k \Psi_i \Psi'_i(-k) = \int_{-\infty}^{+\infty} \phi_i \beta'_i(-k) {\beta'}_i'(-k') \, dk \qquad \text{....(5.3)}$$

substituting (5.2) and (5.3) in (4.9) we have

$$\frac{\partial}{\partial t} \overline{\Psi_i \Psi'_j(k)} + 2 \; \frac{\nu}{P_M} \; k^2 \; \overline{\Psi_i \Psi'_i(k)} = 2i \;\; k_k \; \overline{[\phi_1 \beta'_i {\beta'}_i'}$$

$$- \; \overline{\phi_1 \beta'_i(-k) \;\; \beta_i''(-k')]}_0$$

$$.\exp\left[-\frac{\nu}{P_M} (1+P_M)(k^2+k'^2) + f \frac{P_M}{\nu} + 2P_M k'_k \cos \Theta)\right](t-t_o) \, dk' \text{ .... (5.4)}$$

$$dk' = 2\pi k^2\, d\,(\cos\theta)\, \mathrm{dk}' \qquad \text{....(5.5)}$$

substituing (5.5) to equation (5.4) gives

$$\frac{\partial}{\partial t}\ \overline{\Psi_i \Psi'_j(k)} + 2\ \frac{\upsilon}{P_M}\ k^2\ \overline{\Psi_i \Psi'_i(k)} = 2\int_0^\infty 2\pi i k_k\,[\overline{\phi_1\, \beta'_i \beta'_j{}'}$$

$$-\ \phi_1\, \overline{\beta'_c\,(-k')\ \beta'_i{}'\ \ (-k')\text{-}]_0}\, k'^2$$

$$\times\ [\exp[-\frac{\nu}{P_M}\,(t\text{-}t_o)\,[(1+P_M) + (k^2+k'^2) + P_M K'_k \cos\theta$$

$$+ f\frac{P_M}{\upsilon}]\ d\,(\cos\theta)\,]\,dk' \qquad \text{.... (5.6)}$$

Again following [1] we assume that

$$ik_k\,[\overline{\phi_1\, \beta'_i\ \beta'_i{}'}\ -\ \phi_1\, \overline{\beta'_i(-k)\beta''_i(-k')}]_o$$

$$= -\ \frac{\xi_o}{(2\pi)^2}\ (k^2 k'^4 - k^4 k'^2) \qquad \text{.... (5.7)}$$

Where ξ_o is a constant depending on the initial conditions. From (5.6) and (5.7) we get after integrating with respect to cos θ)

$$\frac{\partial}{\partial t}\ 2\pi\ \overline{\Psi_i \Psi'_j(k)} + 2\ \frac{\nu}{P_M}\ k^2\ \ (2\pi\ \overline{\Psi_i \Psi'_i(k)})$$

$$= -\ \frac{\xi_o}{\nu(t\text{-}t_0)}\ \int_0^\infty (k\ \ k'^s - k^2 k'^3)$$

$$[\exp[-\frac{\nu}{P_M}\ (t\text{-}t_o)\,[(1+P_M)\ .\ (k^2+k'^2) + f\ \frac{P_M}{\nu}\ - 2P_M\, kk']]$$

$$— \exp \left[- \frac{\nu}{P_M} (t-t_o) \left[(1+P_M) + (k^2+k'^2) + f \frac{P_M}{\nu} + 2P_M kk'\right]\right] dk \quad ..(5.8)$$

Multiplying both sides by k^2, we get

$$\frac{\partial H}{\partial t} + \frac{2\nu\ k^2}{P_M} H = G. \quad .... (5.9)$$

Where $H = 2\nu\ k^2 \ \overline{\psi_i \psi'_i (k)}$ is the magnetic energy spectrum function and G is the energy transfer terms given by

$$G = — \frac{\xi o}{(t-t_o)} \int_o^\infty (k^3 * k'^5 - k^5 k'^3)\left[\exp \left(\frac{\nu}{P_M} (t-t_o) \left[1+P_M\right) (k^2+k'^2)\right.\right.$$

$$\left. + f \frac{P_M}{\nu} — 2P_M kk'\right]\Big) —\exp \left(\frac{-\nu}{P_M} (t-t_o) \left[(1+P_M) (k^2+k'^2)\right.\right.$$

$$\left.\left. + 2P_M kk'\right]\right) + f \frac{P_M}{\nu}\Big] \ dk' \quad ....(5.10)$$

Following [1] if we put the value of G is (5.9) we can find H.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 107-112

MOTION OF PARTICLES SUSPENDED IN TURBULENT FLOW

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(Received 8.12.1997)

ABSTRACT

In this paper we have derived the expression for velocity correlation in terms of defining scalars.

Keywords : Isotropic turbulence Reynolds number.

INTRODUCTION

Theoretically very little is known about the fluid dynamics of a mixture of discrete particles of arbitrary size and concentration with a fluid, where both are in turbulent motion. For high concentrations of very fine particles, the mixture may be considered a homogeneous suspension and may be treated is a non Newtonian fluid with certain over-all flow features. For bigger particles, such as grains and sand, Bagnold has studied stress-strain relationships of the mixture, in nearly turbulent motion. An extensive theoretical study has been made by Corrsin [1]. Saffman [2] derived an equation that describes the motion of a fluid containing small dust particles.

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Dixit [3] discussed the rate of change of vorticity covariance in MHD dusty turbulence. In this paper we consider the motion of solid particles with respect to the turbulent gas in which they are suspended. the result obtained can also be applied to the motion of solid particles, underformed droplets and bubbles in a liquid and to the motion of small liquid droplets in a turbulent gas. Here we will consider only the case in which the specific volume occupied by the particles is small compared to unity. The turbulent flow is assumed to be locally homogenous, isotropic and stationary and dimension R of the suspended particles is assumed to be so small that the Reynolds number for the relative motion of the gas is small compared with unity.

EQUATION OF MOTION

The equation of motion of particles was obtained by Corsin and Lumely [1]. This equation relates the velocity W_i of a particle in a gas to the velocity of motion U_i of the gas. In general from it is a non linear particle differential equation. It can however be transformed into a linear integro-differential equation with the time as the only independent variable if

$$\frac{R^2}{\nu}\frac{\partial U}{\partial x} << 1, \frac{W}{\nu}\frac{\partial U}{\partial x} \Big/ \frac{\partial^2 U}{\partial x^2} >>> 1.$$

where ν is the kinematic viscosity of the gas, let U_λ and W_λ be the velocity and frequency of pulsation with dimensions. Using the fact that the particle velocity is close to the velocity U_L of the largest oscillations of the gas, which are of a scale L of order of the dimensions of the system, we have

$$\frac{W}{\nu}\frac{\partial U}{\partial x} \Big/ \frac{\partial^2 U}{\partial x^2} \sim \frac{W}{\nu}\frac{U_\lambda}{\lambda} \Big/ \frac{U_\lambda}{\lambda^2} \sim \frac{U_L \lambda}{\nu} > \frac{U_L}{\nu}\frac{1}{L}$$
$$\sim R_e L^{1/4} >> 1.$$

$$\frac{R^2}{\nu}\;\frac{\partial U}{\partial x} \sim \frac{R^2}{\nu}\;\frac{U_\lambda}{\lambda} \sim \frac{R^2}{\nu}\;w_\lambda < \frac{R^2}{\nu}\;\frac{\nu}{l^2}\;\frac{R^2}{l^2} < 1.$$

Hence, if $R<1$ the equation of steady relative particle motion is

$$\frac{dV_i}{dt} = (\alpha-1)\,\frac{dU_i}{dt} - \beta V_i - \sqrt{\frac{3\alpha\beta}{\pi}}\;\int_{-\infty}^{t}\frac{dV_i}{dt}\;\frac{dT}{\sqrt{t-T}} + g_i \qquad (2.1)$$

$V_i = W_i - U_i$ is the velocity of a particle with respect to the gas,

$\alpha = \dfrac{3\rho_o}{2\rho+\rho_o}$, where ρ_o is the gas density and ρ is the particle density, g_i is the acceleraton of freefall, and $\beta = \dfrac{k_1\alpha\nu}{R^2}$ is the Characteristic frequency determing the motion of the particles in the gas and is equal to the reciprocal of the time required for the gas flowing past a particle to alter its flow state. The quantities k_i are numerical constants of order unity. Since (2.1) as linear, the velocity V_i is the superposition of two independent quantities : the velocity due to gravity and the velocity related to the motion of the gas. Since here we are interested inthe particle motion caused by the movement of the gas, term g_i can be omitted from (2.1). Under our assumption $R<1$, (2.1) can be further siplified. It can be shown that the term involving the integral is small compared to the remaining $\dfrac{w_\lambda R^2}{\nu} << 1$ for all frequencies of the turbulence spectrum .

In fact

$$w_\lambda\,\frac{R^2}{\nu} < \frac{w_i\,R^2}{\nu} \sim \frac{R^2}{l^2}$$

Hence, under our assumptions the equation for the relative motion of a particle in the gas is

$$\frac{dV_i}{dt} = (\alpha\text{-}1)\,\frac{dU_i}{dt} - \beta B_i \tag{2.2}$$

In turbulent flow the velocities U_i and V_i are random functions of the time. To determine the properties of the motion of the particles with respect to the gas we must know the relation between the correlation of U_i and correlation of V_i. Equation (2.2) can be written as

$$\frac{dx_i}{dt} = -\beta V_i \tag{2.3}$$

Where $x_i = V_i = (\alpha\text{-}1)\,U_i$

DISCUSSION OF THE PROBLEM

Let x'_i and V'_i be the values of the variables at time t at point $P(x'_i)$ and X''_j and V''_j be the values of variables at point $P(x''_j)$ then

$$\frac{dx'_i}{dt} = -\beta V'_i \tag{2.3}$$

and $$\frac{dx''_j}{dt} = -\beta V''_j \tag{3.2}$$

From Equation (3.1) and (3.2) we have

$$\frac{d}{dt}\;\overline{x'_i\;\;x''} = \beta^2\;\overline{V'_i}\;\overline{V''_j} \tag{3.4}$$

Let $\overline{x'_i \; x''_j} = P_{ij}$

and $\overline{v'_i \; v''_j} = R_{ij}$

P_{ij} and R_{ij} being isotropic tensors can be written in the from

$$P_{ij} = \alpha(r,t) \, \zeta_i \zeta_j + \beta(r,t) \, \zeta_{ij} \qquad (3.5)$$

$$\text{and} \quad R_{ij} = \frac{R}{r} \zeta_i \zeta_j - (r,R' + 2R) \, \delta_{ij} \qquad (3.6)$$

Putting these in (3.4) me have

$$\frac{\partial}{\partial t} \alpha(r,t) = \beta^2 \, \frac{R}{y}$$

$$\text{and} \quad \frac{\partial}{\partial \mathrm{t}} \beta(r,t) = -\beta^2 \, (rR' + 2R) \qquad (3.8)$$

where $\zeta_{\mathrm{j}} = x''_{\mathrm{i}} - x'_{\mathrm{i}}$

The equations (3.7) and (3.8) give the expression for the rate of change of velocity covariance of particles in terms of defining scalars.

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Journal of Natural & Physical Sciences Vol. 11 (1997) 113-120

INFLUENCE OF AIR POLLUTION ON WEATHER THROUGH SOLAR ACTIVITY

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(Received 15.01.1998)

ABSTRACT

Solar activity affects the ionization in the atmosphere. The atmospheric ions are attached to aerosol particles. These heavy charged particles change the atmospheric conductivity due to change in ion density and mobility. These ions also affect nucleation, formation of rain and lightning activity. Thus these parameters are likely to be strongly correlated with ionization in the atmosphere. Therefore pollution particles should enhanced the type of correlation of precipitation and lightning activity with solar activity. On the basis of above results, we have described the atmospheric processes blockwise, putting one block for each mechanism. Each block may be treated as a system with occasional feedback and dissipation. Using the standard technique of system analysis, one can find the output of each block in exact mathematical terms.

INTRODUCTION

Some atmospheric parameters like precipitation, snowfall, atmospheric electric field, lightning activity, surface temperature, pressure and wind are known to correlate with solar cycles [4].

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Radiations from the sun is the dominant source of energy input to the earth's atmosphere. Sun is not only the primary source of all forms of energies on earth, it is also responsible for wheather phenomenon on our planet. The changes occuring in sun are deemed to affect the weather. A direct link between solar variability and weather changes is indeed a prime factor. The average incoming electromagnetic radiation received from the sun at the top of the atmosphere over all wavelengths is 1353 watt/m^2 [4]. This energy is really constant with time and therefore, it is known as 'Solar Constant'.

It is generally beleived that solar activity modulates the flux of galactic cosmic radiation [2]. This solar modulation takes place as the galactic cosmic rays propagate through the region around the sun containing the interplanetary medium. This interplanetary medium controls the primary galactic cosmic radiations and thereby, the flux of secondary cosmic radiations are generated in the stratosphere [9]. The later is generally responsible for atmospheric ionization.

Cosmic rays are known to create enhanced ionization in the atmosphere. This would increase the conductivity. Increased conductivity decreases atmospheric electrical field. The atmospheric ionization contributes significantly to the charge separation process in convective clouds. During the formation of convective clouds, ions present in the atmosphere are carried with the convection to higher levels where it is surrounded by lower ion density. Hence, it creates stronger field to attract ambient opposite ions to migrate towards cloud boundary. These ions are transported to bottom of the cloud with downward air movements. Naturally, the higher ionization at lower atmosphere will result in stronger charge separation giving enhanced lightning activity [14].

The effect of sun on atmospheric electricity was observed for the first time by Watson [15]. Stringfellow [13] found that the 5 year running mean of annual mean sunspot numbers and the annual index of lightning were in phase for the period 1930 to 1970. Markson [8] explained this correlation in terms of ionizing radiation from sun. An increase in the ionization decreases the atmospheric resistivity, thereby causing decrease in the atmospheric electric field. The change in atmospheric electric field affects the process of charge separation and precipitation.

LATITUDE AND ALTITUDE DEPENDENCE OF COSMIC RAY IONIZATION

The cosmic ray ionization depends upon both the latitudes and altitudes. Manzano and Winckler [7], Shea et al. [12], Nehar [10], Kent and Pomerantz [5], Lockwood et al. [6] and Agarwal [1] studied the cosmic ray ionization with latitudes. Ion production rate by galactic cosmic rays increases from equator to about 50^0-60^0 geomagnetic latitude, where it breaks to form a "knee" [10]. Polweard of the knee, the ion production rate is constant because of decreased magnetic rigidity effects in the nearly vertical geomagnetic field. Nehar [10] proposed that rigidity effects vary through the solar cycle. Ionization by cosmic rays also depends on altitudes. Maximum ion production takes place at altitudes of about 12 to 20 km. The height dependence of cosmic ray ionization has been given by Gish [3] for magnetic latitudes 3^0N and 51^0N. Agarwal [1] showed that the ground level ionization at all latitudes is mainly due to galactic cosmic rays [GCR]. However, at high latitudes the main source of ionization around 20 km and above is due to solar cosmic rays [SCR]. Shea et al. [12] explains the phenomena is terms of Forbush decrease where increased flux of SCR decrease the GCR intensity due to electrostatic shielding of the later.

According to Shea et al.[12] the solar wind protons at high latitudes are dominant over the galactic cosmic rays. Upto about 30, the ion pair production rate is approximately constant, after which it increases very fast. Nehar [10, 11] reported that cosmic ray intensity is minimum at the equator and increases by about two orders of magnitude until 60^0 geomagnetic latitude. After this it becomes constant. From the above we conclude that at low latitudes the atmospheric ionization is mainly due to GCR while at high latitudes it is due to SCR. At mid latitudes from 30^0 to 50^0 contribution is from both sides and it is highly fluctuating.

TIME VARIATION OF LIGHTNING ACTIVITY

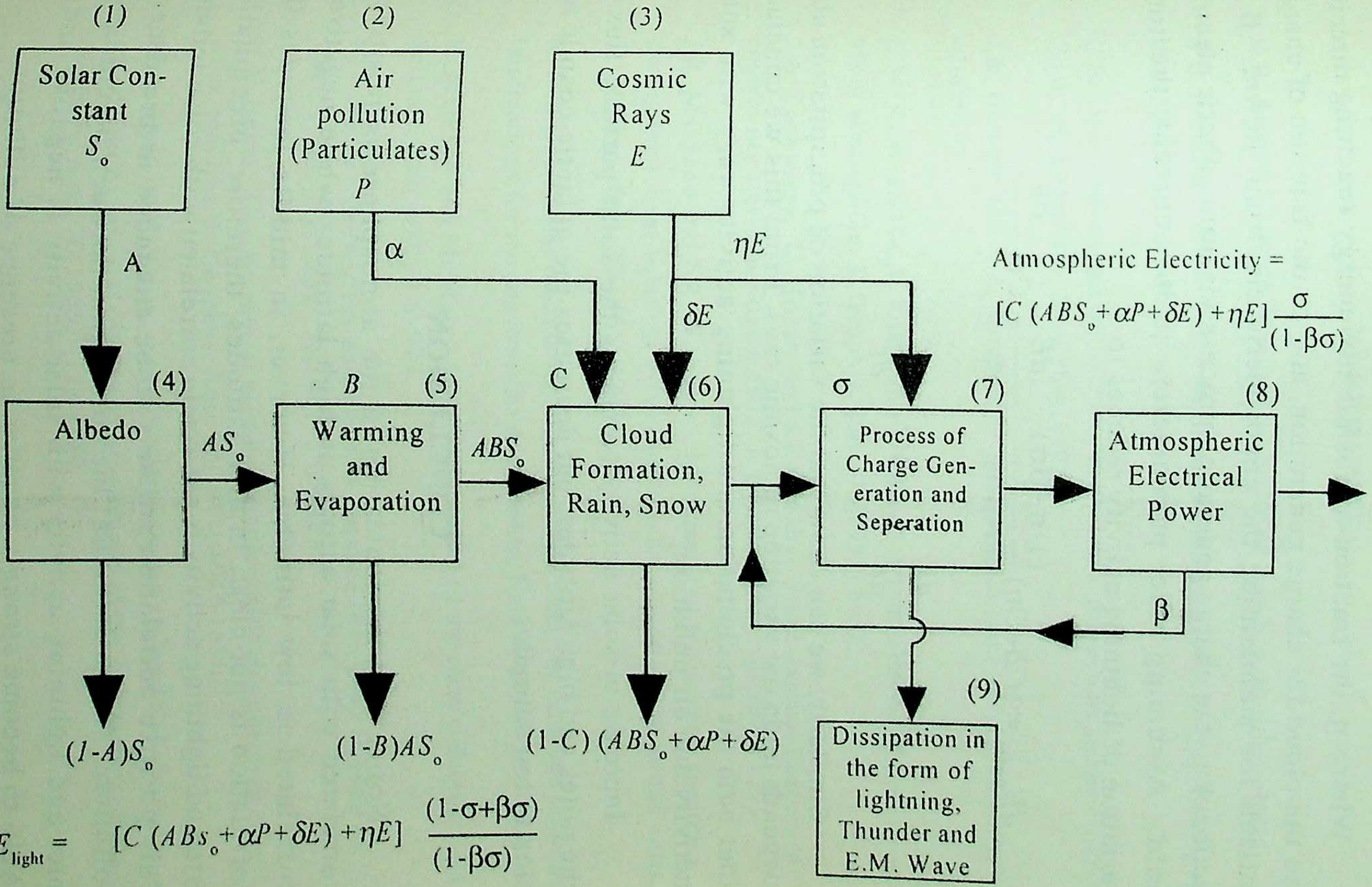
Let S_o be the solar visible and infra-red energy input to the atmosphere with cosmic ray energy E in presence of albedo [which transfers AS_o energy to warming and reflects back $(1-A)\,S_o$ to space]. let P be energy due to air pollution particulates and α be the fraction of this energy utilized in cloud formation then the energy released in form of precipitation is given by,

$$E_{Prec} = (1-C)\,(ABS_o+\alpha P+\delta E) \tag{1}$$

Where B is the fraction of energy which is transferred to cloud formation process. δ is the fraction of cosmic ray energy utilized in the process of nucleation (hence cloud formation), and C is the fraction of energy from cloud transferred to charge generation process. The lightning intensity can be written as (see Fig. 1)

$$E_{\text{lightn}} = [C\{ABS_o+\alpha P+\delta E) + \eta E\}]\frac{(1-\sigma+\beta\sigma)}{(1-\beta\sigma)} \tag{2}$$

Fig. 5.2 - **BLOCK DIAGRAM OF ATMOSPHERIC PROCESSES**


(1)
Solar Constant S_o
(2)
Air pollution (Particulates) P
(3)
Cosmic Rays E
A
α
ηE
δE
(4)
Albedo
AS_o
B
(5)
Warming and Evaporation
ABS_o
C
(6)
Cloud Formation, Rain, Snow
σ
(7)
Process of Charge Generation and Seperation
(8)
Atmospheric Electrical Power
β
(9)
Dissipation in the form of lightning, Thunder and E.M. Wave
$(1-A)S_o$
$(1-B)AS_o$
$(1-C)(ABS_o+\alpha P+\delta E)$

Atmospheric Electricity =

$$[C\ (ABS_o+\alpha P+\delta E)+\eta E]\frac{\sigma}{(1-\beta\sigma)}$$

$$E_{light} = [C\ (ABs_o+\alpha P+\delta E)+\eta E]\ \frac{(1-\sigma+\beta\sigma)}{(1-\beta\sigma)}$$

Where η, the fraction of cosmic ray energy reaching directly into the cloud for charge generation, and σ, the fraction of energy utilized for maintaining the atmospheric electrical power. β is feedback to the charge generation process via atmospheric electric field. Assuming other parameterers to be constant, the time variation of lightning activity reduces to

$$dE_{\text{lightn}} = (C\delta + C\eta)\,\frac{(1-\sigma+\beta\sigma)}{(1-\beta\sigma)}\;\frac{dE}{dt} + C\alpha\,\frac{dp}{dt}$$

$$= \text{Const.}\ \frac{dE}{dt} + \text{const.}\ \frac{dp}{dt} \qquad (3)$$

Similarly, we can show that the variation of precipitation also depends upon the variation of cosmic rays. From this we conclude that both the precipitation and lightning activity vary with solar activity and air pollution.

Increase in solar activity enhances the solar particle flux at the poles and high latitudes, but decreases the galactic cosmic ray flux at low latitudes.

CONCLUSION

Notice that the ionization in the atmosphere is positively correlated with solar activity in high latitudes while negatively correlated in low latitudes. However, in middle latitudes the correlation is not clear. In high latitudes, therefore, precipitation rate and lightning activity is positively correlated with solar activity. This positive correlation is likely to be enhanced in presence of pollution. At low latitudes, the correlation between precipitation rate, and lightning activity with solar activity is negative and is likely to become strongly negative in presence of pollution.

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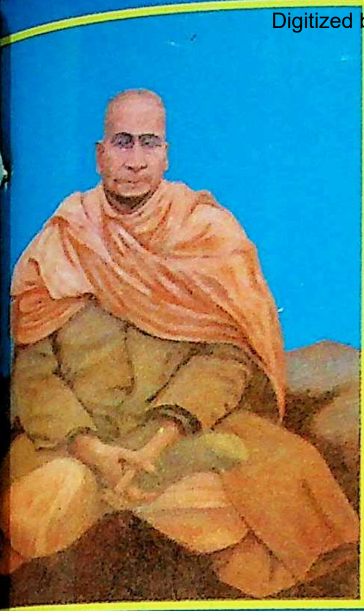

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खण्ड 12, अंक 1, 1998

Journal of Natural & Physical Sciences

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राष्ट्रपति सचिवालय,
राष्ट्रपति भवन,
नई दिल्ली - 110004.
President's Secretariat,
Rashtrapati Bhavan,
New Delhi - 110004.

No.F.2-M/98

30th November, 1998

Dear Dr. Dharm Pal,

The President of India, Shri K.R. Narayanan, is happy to know that the Gurukula Kangri Vishwavidyalaya, Hardwar is organising the 64th Annual Conference of the Indian Mathematical Society (IMS) and a Symposia on "Traditions of Vedic Mathematics & Applications" and "Fixed Point Theory in Nonlinear Analysis & Applications" from 19th to 22nd December, 1998.

The President extends his warm greetings to the participants and the organisers and wishes the Conference and Symposia all success.

With regards,

Yours sincerely,

Satya Narayana
(S.N. Sahu)

Dr. Dharm Pal,
Chairman, Reception Committee &
Vice-Chancellor,
Gurukula Kangri Vishwavidyalaya,
Hardwar 249 404.

OM

Gurukula Kangri Vishwavidyalaya, Hardwar

Prof. Sher Singh
Visitor

☎ 011 - 685171
011 - 685923
011 - 685771
Fax : 011 - 652252

M-14, Saket
New Delhi 110 017

Date 10-12-98

Dear Dr. Dharam Pal:

I am happy to know that a symposia on Traditions of Vedic Mathematics an Applications is being organized during the 64th IMS Conference (19-22 Dec 1998 at Gurukula Kangri Vishwavidyalaya, Hardwar. The IMS is a premier academi body of India dedicated to the cause of Mathematics & Mathematics Education Therefore its conference is of valuable importance to the society and mankind.

I wish very fruitful deliberations of the conference and symposia on Vedi Mathematics and Fixed Point Theory.

Sincerely

Professor Sher Sing
Visitor, GKV &
Chairman, International Dayanand Ved Peet

Dr. Dharam Pal
Chairman, Reception Committee, 64th IMS Conf. &
Vice-Chancellor, Gurukula Kangri Vishwavidyalaya
Hardwar 249 404

OM

Gurukula Kangri Vishwavidyalaya, Hardwar

Surya Dev
Chancellor

Phone : 3264129
1862, Cheerakhana
Maliwara, Delhi 110 006

Date : 12-12-98

Dear Dr. Dharam Pal:

I am happy to know that the 64th Annual Conference of the Indian Mathematical Society is being organized in Gurukula Kangri Vishwavidyalaya during Dec. 19-22, 1998 under your dynamic leadership. Perhaps, it is the first time that the Vishwavidyalaya is organizing such a conference of vital importance in Science College. Further it gives me immense pleasure to know that Professor S. L. Singh, Principal Science College and the Local Secretary of the conference is also organizing a symposium on Vedic Mathematics. Vedic sciences are an integral part of Indian culture and traditions. So the vision behind organizing the symposia and conference in GKV is highly compatible to the vision of the founder of the GKV Svami Shradhanand Ji Maharaj.

I send my greetings and good wishes for a grand success of the conference and symposia.

Sincerely,

सूर्यदेव

Surya Dev
Chancellor, Gurukula Kangri Vishwavidyalaya

Dr. Dharam Pal
Chairman, Reception Committee, 64th IMS Conf. &
Vice-Chancellor, Gurukula Kangri Vishwavidyalaya
Hardwar 249 404

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Journal of Natural & Physical Sciences Vol. 12 (1998), No. 1, 1-3

स्वागत – भाषणम्

डॉ0 धर्मपालः
स्वागताध्यक्षः कुलपतिश्च,
गुरुकुल कांगड़ी विश्वविद्यालयः हरिद्वारम्

अमरहुतात्मना स्वामिश्रद्धानन्देन संस्थापिते गुरुकुलकांगड़ीविश्वविद्यालयस्य गणितविभागे समायोज्यमाने गणितविद्वत्सम्मेलने गुरुकुलभुवमागता मान्याः कुलाधिपतयः श्रीमन्तः सूर्यदेवमहोदयाः, मुख्यातिथयः, भारतीय-गणित-परिषदो मान्या अध्यक्षाः प्रो0 डॉ0 बी.के. लहिरी महोदयाः, परिषदः सचिवाः प्रो0 एच.सी. खरे महोदयाः, आचार्यपदे शोभमानाः प्रो0 वेदप्रकाशशास्त्रिणः, सर्वे संकायाध्यक्षा, विभागाध्यक्षा, प्राध्यपकाः, कुलसचिवाः प्रो0 एस.एन. सिंह महोदयाः, भारतीयविश्वविद्यालयेभ्यः शोधसंस्थानेभ्यस्तथा दूरदेशीयविश्वविद्यालयेभ्यः समागताः गणितविद्यापारङ्गता विद्वांसः, सम्मेलनस्य निदेशका विज्ञानमहाविद्यालयस्य प्राचार्याः प्रो. एस. एल. सिंह महोदयाः, सर्वे विद्वांसश्च।

अद्य युगपदेव समागतान् भवतो विलोक्य सर्वेषां गुरुकुलवासिनां मनांसि सानन्दानि सन्ति। गुरुकुले प्रारम्भकालादेव विज्ञानविषयाणां शिक्षाव्यवस्था समुचिता आसीत्। सम्प्रति विश्वविद्यालये अस्मिन् सप्तसंकायाः सन्ति। आधुनिकविज्ञानविषया अत्र वेदानां परिप्रेक्ष्ये पाठ्यन्ते। अत्र गणितविभागे वैदिकगणितस्यैकं प्रश्नपत्रमध्याप्यते। अत्रत्याः प्राध्यापकाः प्राचीनशास्त्रेषु गणितविद्यामन्विष्य शोधगोष्ठीषु स्वशोधपत्राणि पठन्ति।

अस्माकं कृते प्रमोदावहः सुदिवसो यदद्य सर्वेऽपि विद्वांसो भवन्तो गणितविषये विभिन्नान् विचारान् प्रस्तोष्यन्ति विदुषां मनसि गणितविद्यानुरागं च जनयिष्यन्ति।

अङ्कगणितप्रसङ्गे वेदे विद्यमानं गणितबीजं मया यल्लब्धं तदेव किञ्चिद् प्रकाश्यते।

वेदविद्यारहस्यविदा महिमावता महर्षिणा देवदयानन्देन पूर्वपूर्वतमानामृषीणां सन्देशमुपदेशं च देवगिरा प्रवदता स्पष्टमेवोदीरितम्। यत् समस्तानां सत्यविद्यानां मूलं वेद एव। अर्थात् मानववेद्या या याः सत्यविद्याः सन्ति ताः सर्वा वेदादेव निःसरन्ति। मनुना स्पष्टमेव निर्दिष्टं यत् “भूतं भव्यं भविष्यं च सर्वं वेदात्प्रसिध्यति”। अर्थात् यत् किञ्चिदतीतं यद् वर्तमानं यच्च भविष्यं तत्सर्वं वेदादेव प्रसिध्यति। एतेन ज्ञायते यद् वेदानां सर्वविद्यामूलत्वे नास्ति संशयकणिकोन्मेषावसरः।

अथर्ववेदे गणितस्य विषये सम्प्राप्तये यो मन्त्रस्तत्र त्रिधा गणना कृता। एका संख्या अन्यया संख्यया मिलित्वा का भवति। इत्यस्मिन् प्रश्ने समुद्गते सति एका च दश च मिलित्वा एकादश। यद्येका दश गुणिता भवति तर्हि दश। अस्मिन्नेव मन्त्रे चैकादश संख्याया द्विगुणत्वं त्रिगुणत्वं चतुर्गुणत्वं पञ्चगुणत्वं षड्गुणत्वं सप्तगुणत्वं अष्टगुणत्वं नवगुणत्वं दशगुणत्वं सहस्रगुणत्वं च द्रष्टुं शक्यते।

अत्रमूलरूप एव प्रस्तूयते मन्त्रः –

एका च मे दश च मे।	1+10=11/11X1 = 11
द्वे च मे विंशातिश्च मे।	2+20=22/11X2 = 22
त्रिस्रश्च मे त्रिंशच्च मे।	3+30=33/11X3 = 33
चतस्रश्च मे चत्वारिंशच्च मे।	4+40=44/11X4 = 44
पञ्च च मे पञ्चाशच्च मे।	5+50=55/11X5 = 55
षट् च मे षष्टिश्च मे।	6+60=66/11X6 = 66
सप्त च मे सप्ततिश्च मे।	7+70=77/11X7 = 77
अष्ट च मे अशीतिश्च मे।	8+80=88/11X8 = 88
नव च मे नवतिश्च मे।	9+90=99/11X9 = 99
दश च मे शतं च मे।	10+100=110/11X10 = 110
शतं च मे सहस्रं च मे।	100+1000=1100/11X100 = 1100

अस्मिन्नेव मन्त्रे निर्दिष्टासु सर्वासु पूर्वपठितासु संख्यासु दशधा गुणीकृतासु तृतीयः क्रमः प्रज्ञातुं शक्यते। यथा

1X10 = 10
2X10 = 20
3X10 = 30
4X10 = 40
5X10 = 50
6X10 = 60
7X10 = 70
8X10 = 80
9X10 = 90
10X10 = 100
100X100 = 10,000

एवमेकस्मिन्नेव मन्त्रे त्रिधात्वं विलोक्य विद्यालये यद् गणितविद्यामूलं वेदे विद्यते। संख्या विषये महीयसी परिगणना संस्कृते विद्यते। प्रथमसंख्यातः समारभ्योत्तरोत्तरं दशधा बुद्धिमुपगता संख्या परार्द्धयं यावत् प्रयाति। यथा

एकं दश शतं चैव सहस्रमयुतं तथा।
लक्षं च नियुतं चैव कोटिरर्बुदमेव च।।
वृन्द खर्वो निरवर्वश्च शङ्खः पद्मं च सागरः।

सूर्यसिद्धान्त विषये अनया संख्यया वर्षादिपरिगणना क्रियते। वेदेषु वेदसम्मतेषु च ग्रन्थेषु गणितविद्या समन्वेष्टव्या गणितविद्यावतां धुरिकीर्तनीयैर्विद्वद्भिः। प्रो0 एस. एल. सिंह महोदयैर्निगदितं यत् सम्मेलनेऽस्मिन् बहूनि शोधपत्राणि वेदसम्मतानि सन्ति इदं वृत्तं भारतीयानां सुखकरं विद्यते।

अन्ते चाहं सर्वेषां मान्यानां वदान्यानां स्वागतम् अभिनन्दनं च व्याहरन् अवाच्यमानन्दमनुभवामि। समेषां सुखबृद्धिं यशोबृद्धिं वयोबृद्धिं वित्तबृद्धिं च भूयो भूयः कामये।

नमस्ते भगवन्नस्तु (सामवेदः)

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FROM THE LOCAL SECRETARY — *S. L. Singh*

This is an overwhelming occasion for our Gurukula family that we are organizing such a big conference for the first time in the history of our Science College, that too on a short time of $3\frac{1}{2}$ months, besides our heavy limitations. The deliberations will encompass frontier areas of mathematics and Vedic mathematics which deals with the golden tradition of Ancient Indian Mathematics. Additional happiness stems from the fact that this conference is very much in tune to the mission of this Vishwavidyalaya as visioned by the founder **Svami Shradhanand Ji Maharaj** (1856-1926). We are highly indebted to the IMS President Professor **B. K. Lahiri**, the IMS council members and especially Professor **H. C. Khare** for giving an opportunity to our Vishwavidyalaya to organize this 64th IMS conference, and further consenting to organize two symposia on Vedic Mathematics & Fixed Point Theory.

Now, our delegates are also aware of our difficulties and limitations. Whatever we have been able to arrange is far less than what our guests and delegates deserve. They are in a city famous for *ashrams* and saints, and we request them to have a saintly way of life for a few days. Mathematicians are known for their dedication to their subject and minimal requirements. They are best known to improve known theories by reducing the requirements, any way. The organizing committee extends a very warm welcome to our distinguished guests, the IMS President, past Presidents, council members and all delegates from India and abroad. The success of our organization depends on your kindness and generosity to embrace upon difficulties that they might have to face here. Another dimension of the success is undoubtedly their active participation is the deliberations of the conference and symposia.

As for as I remember, it was the second week of August 1998, when Professor **H. C. Khare** asked me to organize the present IMS conference. Besides the Vishwavidyalaya Calendar and my own programmes incompatible to undertake this penrose work, I started hurried consultations with my departmental colleagues and officers of the Vishwavidyalaya. We were finding only two days (Dec. 21 & 22) for the conference. However, our Vice-Chancellor **Dr. Dharmpal Ji** was enthusiastically in favour of this conference, and, now, I know that he had to sacrifice some of his already fixed schedules too (such as Vice-Chancellors' conference at Indore) and schedule several others. I have been encroaching upon his valuable time during his tight schedules, and it is difficult to find suitable words to appreciate his interest & enthusiasm and express my indebtedness to him for taking enough interest, extending all possible cooperation and maintaining his meaningfull blessings.

Hardwar is a small city and our Vishwavidyalaya is still smaller but both are blessed with good saints and generous persons. The first outside assurance of financial assistance came from Professor **Sher Singh**, Chairman, International Dayanand Ved Peeth, New Delhi and, now, Vishwavidyalaya's Visitor for organizing the conference and symposia on Vedic Mathematics. Although a true Gandhian Prof. Singh has held high positions like central ministership, membership of Planning Commission and currently he is chairman, All India Prohibition Council but this a pleasant coincidence that he is basically a mathematician.

The announcement of the conference could be made in late September' 98 only after the assurance of providing accommodation in Shri Gujrat Dharamshala by Shri Kaku Bhai on a strong recommendation by Shri Ramesh Bhai Thakur of Aryaniwas (Hardwar).

I do not intend to express travails of organizing the conference experienced by some of my departmental colleagues and me during the last 3 months, but I am very happy to express my heartfelt indebtedness to my colleagues, friends, saints and saintly persons from Vishwavidyalaya and outside Vishwavidyalaya, especially from Hardwar & Rishikesh. With the risk of missing many names (from off-campus), I respectfully acknowledge assistance, cooperation and blessings from (in chronological

order of contacting them) : Dr. **Raj Kumar Singh Rawat**, General Manager, Gurukula Kangri Pharmacy; Dr. **Pranav Pandya**, Director, Brahmavarchas Research Institute, Shantikunj; Anant Shri Vibhusit **Rishikeshvanand Ji Maharaj**, Nirdhan Niketan; Shri **Brahma Swarup Brahmachari**, Sanchalak, Shri Jairam Ashram; Svami **Chidanand Ji Muni**, President, Paramarth Niketan (Rishikesh); Svami **Satyamitranand Ji Maharaj**, Ex-Shankaracharya & Founder Bharat Mata Mandir; Professor **R. Anand**, Managing Director, Altek Devices (Delhi); Shri **Motilal Punjwani**, Managing Director, Hotel Midtown. Further valuable cooperation and assistance extended by Shri **I. D. Joshi** (Principal) and Shri **D. K. Varshney** (Math. Lecturer) of Rishikesh; Dr. **Satish Kumari** (Principal) of Nirdhan Niketan; Shri **N. P. Singh** (City Magistrate, Hardwar); Shri **I. D. Shastri** (Bharat Sadan); Shri **N. D. Malhotra**, Manager, Mandir Mata Vaishno Devi, Hardwar; Shri **Girish Chandra Punetha**, Manager, Pawan Dham, Hardwar; Shri **Chandra Kiran Saini**, Kiran Printing Press, Hardwar; Shri Sunil Gupta, BHEL-Hardwar; Meera **Computers** (Delhi); **Kumar Woods** (Jwalapur) and several others are worth mentioning.

The Organizing Committee thanks the CSIR, New Delhi & International Dayanand Ved Peeth, New Delhi for their financial assistance.

Members of the Organizing Committee have played their roles with full dedication and best of their capability.

Teachers and nonteaching staff, whom we approached, extended their full cooperation and help in this venture. The registery and finance department of our Vishwavidyalaya deserves appreciation for not applying much brakes.

Once again I extend very warm greetings to distinguished guests delegetes and others associated with this conference.

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MATHEMATICS EDUCATION*

B. K. Lahiri**

President, Indian Mathematical Society

Distinguished mathematicians, revered members of the Indian Mathematical Society, delegates, ladies and gentlemen.

It is my great honour and distinction to get this opportunity to address a gathering of mathematical luminaries of our country and abroad. I express my hearty gratitude to the office-bearers of the Indian Mathematical Society for electing me to the prestigious position of the office of the President of the premier Mathematical Society like Indian Mathematical Society. I am extremely honoured by their decision and shall endeavour to keep the upward trend of effective academic activities set up by the past office-bearers and members of the Indian Mathematical Society.

I had the privilege of attending for the first time the annual conference of the Indian Mathematical Society in 1955 held in Cuttack. As far as I remember, at that time the academic activities of the Society throughout the year were confined only to the presentation of papers and some special lectures delivered in the annual conference and there were no other activities in between the annual conferences. But during the last several years, apart from the usual activities in the annual conference, the Society implemented several academic programmes such as special lectures, Memorial lectures etc. throughout the year. Moreover, IMS prizes, AMU prize etc. have been instituted and these are awarded for the best research papers in specified areas presented in the conference. Several prizes after the names of great mathematicians like P.L. Bhatnagar, V.M. Shah etc. are also awarded. Moreover, one of the objects of the Society is to inspire and enthuse young researchers of the country so that they may take the responsibility to maintain the present standard of research outputs, which is at par with the international standard. The credit for all these activities go to the successive office-bearers of the Society who meticulously planned and implemented these activities to enliven the Society throughout the year.

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*PRESIDENTIAL ADDRESS (GENERAL), 64th Conference of the Indian Mathematical Society (19-22 Dec. 1998) Gurukula Kangri Vishwavidyalaya, Hardwar

** B-1/146 Kalyani, West Bengal 741235 India

It is customary for the President to make some enlightened remark in the conference on the present day mathematics education and research in the country, pointing out defects, if any with possible remedies. However, this topic has been widely discussed during the last two decades by experts and more experienced persons, some of them are in the audience. Therefore, I am afraid to venture on the topic in the presence of so many specialist whose consistent effort explored many sides and who made suggestions for the positive development of mathematics education and research in our country.

Since the President is to some extent duty bound, I attempt to make remark in short only on post graduate education and research.

After graduation, a considerable number of students enquire employment and most of them having failed to secure proper employment turned their attention for admission to post graduate courses. Even after admission, they keep always an eye for securing suitable jobs and as such, for them admission become a stop-gap arrangement. Consequently, there is a dearth of really good, sincere and devotee students in the fields of higher education and the present socio-economic situations in the country is a major factor for this trend.

Due to the large number of intending candidates, holding of admission test in every institutions may be difficult, screening examinations may be held periodically and then perhaps a considerable portion of undesirable infiltration to the higher studies may be curbed.

Presently, the inclination among the students for research works has been increased considerably without perhaps an increase in number of real admirers for research works. This is supported by the fact that many Ph.D's, after obtaining the degree refrained from producing any research paper for a considerable period. This is happening perhaps due to the following causes :

(i) requirement of research papers or Ph.D. degree for undergraduate college teachers (formulated by the University Grants Commission).

(ii) such Ph.D.'s think that their goal to obtain the degree is achieved.

(iii) such Ph.D.'s are unable to adapt themselves with the training they receive while working for the Ph.D. degree so that the capacity of carrying on researches independently is not grown.

My experience tells that most of the Ph.D. thesis contain little basic and original results. These contain only generalizations of known results having rare indication of significant and fruitful applications of generalized results. I believe that in a Ph.D. thesis a considerable portion which are original in its true sense should be included.

There is no dearth of talented students in the country and so purely original work is very much possible if the persons concerned are afforded with proper opportunity and facilities. But nowadays due to continuous escalation of prices of research journals and mathematical reviews (MR), even the libraries of the affluent institutions are unable to purchase the same. In the case of reprints, Indian National Scientific Documentation Centre (INSDOC) is providing a marvellous service to the users. But even then they sometimes require more than two months time to provide with the reprints and by this there comes the possibility to loose the trace of the main research problem due to the time gap.

There are several mathematical societies in India almost spread over the country and I believe, any worker is a member of one or more of these societies. We urge upon the Department of Atomic Energy, Government of India through the National Board of Higher Mathematics (NBHM) to sanction proper grants to these societies (the number may not exceed ten), except those who are already purchasing MR, to enable them to subscribe MR annually so that the members of these societies, together they cover most of the research workers of the country, may have regular access to the MR.

Thanking again all of you, I conclude my brief address by wishing a meaningful success of the conference so that the remnant continues for days to come.

Journal of Natural & Physical Sciences Vol. 12 (1998), No. 1, 10-23

NORMAL STRUCTURE*

B. K. Lahiri**

President, Indian Mathematical Society

The definition of normal structure was introduced by Brödskii and Milman, primarily to solve some geometrical problems. In fact, at a later stage a constant has been defined to be the supremum taken with respect to the convex bounded subsets C of a Banach space X, of the ratio between the Chebyshev radius of C relative to C, and the diameter of C, which has close connection with normal structure. However, besides serious subsequent theoretical investigations on this concept of Brödskii and Milman and related ideas developed later on, the concept of normal structure becomes dominant in the theory of fixed points of operators, specially for nonexpansive mappings. If $T : X \to X$ is a mapping where X is a Banach space then T is called nonexpansive if $\| Tx-Ty \| \leq \| x-y \|$ for all $x,y \in X$. A closed convex subset C of a Banach space X is said to have the *fixed point property* if every nonexpansive $T : C \to C$ has a fixed point.

Here we provide only a glimpse of the idea of normal structure, the presentation, however, could not be exhaustive because of the enormity of the investigations made in this topic recently. We have not touched upon the topics on variants of normal structure, normal structure in spaces besides Banach spaces, sum property, quotient spaces relating to normal structure etc. Unless otherwise stated, X stands for a Banach space.

DEFINITIONS AND EXAMPLES

The following definition was introduced by Brödskii and Milman [5].

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Definition 1 : A convex set $K \subset X$ is said to have *normal structure* if for each bounded closed convex subset H of K which contains more than one point, there is some point $x \in H$ such that

$$\sup\{\| y-x \| : y \in H\} < \text{diam}\,(H).$$

An equivalent definition has been given by themselves as follows :

Difinition 2 : X has normal structure if and only if there exists in X no bounded non-constant sequence $\{x_n\}$ such that

$$d(x_n, \text{co}\{x_j\}_{j=1}^{n=1})$$

i.e. the distance from the nth element of the sequence to the convex hull of the preceding elements, approaches the diameter of the sequence as $n \to \infty$.

The chebvshev radius of a closed convex subset A of X, denoted by $R(A)$, is defined to be

$$\inf\{\sup\,(\|y-x\| : y \in A), x \in A\}.$$

It is clear that X has normal structure if $R(A) < \text{diam}\,(A)$ [22] for every non-singleton closed bounded convex subset A of X.

Definition 3 : Let K denote the unit sphere in X and let $\delta(t)$ be defined in [0, 2] as follows :

$$2\delta(t) = \inf\{2 - \| x+y \| : x, y \in K, \| x-y \| \geq t\}.$$

The function $\delta(t)$ is known [19, 7] to be *modulus of convexity* and X is *uniformly convex* if $\delta(t) > 0$ for $t \in (0,2]$ {[9, 19] }. The *coefficient of convexity* [12] of X, $\varepsilon_0 = \varepsilon_0(X)$ is

$$\varepsilon_0(X) = \sup \{t \in [0, 2] : \delta(t) = 0\}.$$

Example 1 [6] : For $1 < p < \infty$, let $x = \{x_n\} \in l_p$. Let x^+ and x^- be sequences defined as

$$x^+ = \{x_n^+\} \text{ where } \quad x_n^+ = \max\{x_n, 0\}$$

and $$x^- = (-x)^+.$$

Clearly x^+ and $x^- \in l_p$. If $\|\cdot\|$ denotes the norm in l_p and $1 \le q < \infty$, let $l_{p,q}$ denote the set l_p with the norm

$$|x| = (\|x^+\|^q + \|x^-\|^q)^{1/q}$$

and let $l_{p,\infty}$ be the set l_p with the norm

$$|x| = \sup\{\|x^+\|, \|x^-\|\}.$$

Then $|\cdot|$ is a norm for l_p which is equivalent to the l_p norm. $l_{p,\infty}$ lacks normal structure, $\varepsilon_0(l_{p,\infty}) = 1$, while $l_{p,1}$ has normal structure and

$$2^{1/p} \le \varepsilon_0(l_{p,1}) < 2.$$

Example 2 [18] . Non-void compact convex sets have normal structure. Uniformly convex Banach space has normal structure. The spaces $C[0,1]$, L^1, L^∞ lack normal structure.

Example 3 [24] . In every Banach space an equivalent norm can be introduced so as to have normal structure.

We observe a lack of connection between normal structure and reflexivity of the space. In [15] the authors gave an example of a reflexive Banach space lacking normal structure. So reflexivity does not imply normal structure. On the other hand, Zizler [24] proved that each separable Banach space X has an equivalent norm with respect to which X has normal structure. This implies that normal structure does not imply reflexivity.

NORMAL STRUCTURE COEFFICIENTS

Characterization of a Banach space X to have normal structure has been determined in terms of coefficient of convexity and in fact it is known that if the coefficient of convexity is less than one then X has normal structure [12]. The converse is not true as is evident from the space $1_{p,1}$ (Ex 1). Also there is an example of a class of spaces which lack normal structure and have coefficient of convexity one [6] (for example, the space $1_{p,\infty}$, Ex 1). Another important direction is to determine the normal structure coefficients for Banach spaces and in terms of these to determine if a Banach space has normal structure. We proceed to define the coefficients.

Definition 4 [7]: $N(X) = \inf D(C) \,/\, R(C)$

where the infimum is taken over all bounded closed convex subsets C of X with more than one member.

Here $D(C)$ denotes the diameter of C and $R(C)$, the Chebyšev radius of C.

Definition 5 [7]. Let $\{x_n\}$ be a bounded sequence in X. The expression

$$\lim_{n} [\, \sup \{ \| x_m - x_k \| : m \geq n, k \geq n \}]$$

is called the *asymptotic diameter* of the sequence $\{x_n\}$.

Definition 6 [7] : The *bounded sequence coefficient* of X, denoted by $BS(X)$, is the supremum of the set of all numbers M such that for each bounded sequence $\{x_n\}$ with asymptotic diameter A, there is some y in the closed convex hull of the sequence such that

$$M.\ \limsup_{n}\ \| x_n - y \| \leq A.$$

Definition 7 [7] : *The weakly convergent sequence coefficient* of X, denoted by $WCS(X)$ is defined as in Definition 6 replacing "bounded" by "weakly convergent".

These three coefficients are connected by the following relation [7]

$$1 \leq N(X) \leq BS(X) \leq WCS(X) \leq 2 \qquad \ldots(1)$$

provided X is reflexive.

The coefficients $N(X)$, $BS(X)$ and $WCS(X)$ are sometimes called Banach space coefficients or normal structure coefficients.

For certain spaces it has been possible to calculate the above coefficients. For example, if $1 < p < \infty$ then $WCS(l_p) = 2^{1/p}$ [7] and $N(l_2) = 2^{1/2}$ [10]. So we observe that for the space l_2, all the three constants coincide. Also, if any one of these coefficients is greater than one, then X has normal structure [7]. However, there are spaces where all the constants are one although the space has normal structure. For example, let X be the l_2- direct sum of the sequence of spaces $\{l_n\}$, $n \geq 2$, i.e. let X be the collection of all sequences $x = \{x_n\}$, $n \geq 2$ such that x_n is in l_n for each $n \geq 2$ and

$$|x|^2 = \sum_{n} \| x_n \|^2 < \infty.$$

This space has normal structure but all the three constants for this space is one.

It has been proved in [7] that if X is reflexive with modulus of convexity δ, then

$$N(X) \geq (1\text{-}\delta(1))^{-1}.$$

So if $\delta(1) > 0$ then $N(X) > 1$ and X has normal structure.

ASYMPTOTIC NORMAL STRUCTURE

The concept of asymptotic normal structure was introduced by Schöneberg and coworkers [4, 21] in connection with the theorem on the existence of fixed points of a nonexpansive selfmap of closed convex bounded subsets of a Banach space.

Definition 8. A Banach space X has *asymptotic normal structure* if for each closed bounded convex subset C of X with more than one member and each sequence $\{x_n\}$ in C for which $\| x_n\text{-}x_{n+1} \| \to 0$ as $n \to \infty$, there is a point x in C such that

$$\liminf_{n} \| x_n\text{-}x \| < \text{diam}\,(C).$$

Clearly normal structure implies asymptotic normal structure, but the converse is not true.

For $\beta > 1$, let E_β be the space l_2 renormed as

$$|x| = \max\,\{\, \| x \|_2,\ \beta \| x \|_\infty \}$$

where $\| x \|_\infty = \sup\limits_{n}\ \{\, | x_n | \,\} < \infty$ is the l_∞ norm and

$\| x \|_2 = (\sum_{n=1}^{\infty} |x_n|^2)^{\frac{1}{2}}$ is the l_2-norm. Baillon and Schöneberg [4] showed that E_β has asymptotic normal structure if and only if $\beta < 2$ and has normal structure if and only if $\beta < \sqrt{2}$. Thus if $\sqrt{2} \leq \beta < 2$ then E_β has asymtomatic normal structure without having normal structure.

A question was raised if there exists a reflexive Banach space without asymptotic normal structure. We consider the space $l_{p,\infty}$ with the norm $|x|$ considered in section 2. A short computation shows that for all x in l_p $(1 < p < \infty)$

$$|x| \leq \|x\| \leq 2^{1/p}|x| .$$

Since l_p is reflexive, this inequality shows that $l_{p,\infty}$ is reflexive. The space $l_{p,\infty}$ lacks asymptotic normal structure [8] .

We proved [16] a general theorem in this direction by showing that in every infinite dimensional separable Hilbert space an equivalent norm can be introduced for which it lacks asymptotic normal structure.

NORMAL STRUCTURE UNDER ISOMETRY

If X and Y are isomorphic Banach spaces then the *Banach - Mazur distance* from X to Y, denoted by $d(X,Y)$ is

$$d(X,Y) = \inf \|U\| \, \|U^{-1}\|$$

where the infimum is taken over all bicontinuous linear operators U from X onto Y. The distance function $d(X,Y)$ is connected with the above three constants in the following way [7] :

$$N(X) \leq d(X,Y)N(Y),$$
$$BS(X) \leq d(X,Y)BS(Y),$$
and $$WCS(X) \leq d(X,Y)WCS(Y).$$

It is known [17] that if X and Y are linearly isometric then X has normal structure or asymptotic normal structure if and only if Y has so and that

$$N(X) = N(Y), BS(X) = BS(Y), WCS(X) = WCS(Y), \qquad \ldots(2)$$

The equalities (2) are sometimes very helpful to determine the normal structure coefficients for some spaces.

Example 4. Bynum [7] remarked that $N(L_p\,[0,1])$, $1 < p < \infty$ is difficult to determine and no precise value could be obtained. Recently Prus [20] showed that

$$N(L_p\,[0,1]) = WCS\ (L_p\,[0,1]) = \min\{2^{1/p}, 2^{1/q}\},\ 1/p + 1/q = 1,\ p > 1,$$

but Prus' deduction is too involved. As an application of (2), we can immediately obtain the value of $N(L_2\,[0,1])$. Because $N(l_2) = 2^{1/2}$ {see [7] } and $L_2\,[0,1]$ is linearly isometric to l_2, so $N(L_2\,[0,1]) = 2^{1/2}$.

Example 5. Using (1) and the result of Prus, we see that $BS(L_2[0,1]) = 2^{1/2}$.

Example 6. We consider the space R^n_∞ $(n \geq 1)$, where R^n_∞ is the space R_n $(n \geq 1)$ with the maximum norm. Let X be the 1_2-direct sum of the space $R^n_\infty\,(n \geq 1)$, then it is known [7] that

$$N(X) = BS(X) = 1,\ WCS(X) = 2^{1/2}$$

and also $N(l_2) = BS(l_2) = WCS\ (l_2) = 2^{1/2}$ {see [7] }.

So from (2) it follows that X and 1_2 are not linearly isometric, although one of the coefficients, viz. $WCS\ (X)$ is the same for the spaces.

APPLICATIONS OF NORMAL STRUCTURE

A mapping $T: X \rightarrow X$ is called non-expansive if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in X$. A nonexpansive mapping may not have a fixed point [2] . Using normal structure, Kirk [13] proved the following basic theorem.

Theorem 1. If X is a reflexive Banach space having normal structure and C is a closed convex bounded subset of X and $T : C \to C$ is nonexpansive then T has a fixed point in C.

It was an open question till 1976 if the normal structure requirement in the theorem was essential. In 1976 Karlovitz [13] considered a space X which is l_2 -renormed as

$$\| x \| = \max \{ \| x \|_\infty , \| x \|_2 / \sqrt{2} \}.$$

He considered the subset

$$C = \{ x \in X : x = \{x_i\}, x_i \geq 0, \| x \|_2 \leq 1 \}$$

and proved that it has no w for which

$$\sup \{ \| w - y \| : y \in C \} < \operatorname{diam} (C)$$

and so X does not have normal structure. But he showed that there exists $T : C \to C$ which is nonexpansive and has a fixed point in C.

Achari and Lahiri [1] proved a fixed point theorem for a mapping $T : C \to C$ where C is a nonempty closed convex bounded subset of a reflexive Banach space X having normal structure where T satisfies the conditions

$$\text{(i)} \quad \| Tx - Ty \| \leq \max\{ \| x - y \| , \| x - Tx \| , \| y - Ty \| \}$$

for all $x, y \in C$;

$$\text{(ii)} \sup_{y \in F} \| y - Ty \| \leq \tfrac{1}{2} \operatorname{diam} (F)$$

for every non-empty closed convex subset F of C, containing more than one element and mapped into itself by T.

Tiwary and Lahiri [23] relaxed condition (ii) above as

$$\text{(ii)} \sup_{y \in F} \| y - Ty \| < \operatorname{diam}(F)$$

and proved the same theorem.

Baillon and Schöneberg [4] proved the above fixed point theorem of Kirk replacing normal structure by asymptotic normal structure.

If X and Y be isomorphic Banach spaces and $d(X,Y)$ be the Banach-Mazur distance from X to Y, Bynum [7] proved that if X is uniformly convex and Y be such that $d(X,Y) \le WCS(X)$ then each nonexpansive self-map of a closed bounded convex subset of Y has a fixed point.

If X is uniformly convex, Prus [20] defined a constant $\alpha(X)$ by

$$\alpha(X) = \sup \{ \alpha > 1 : \alpha ((1 - \delta(2/\alpha)) \frac{WCS(X)}{2} + 1 - \delta(1/\alpha)) < 1 \}$$

where δ is the modulus of convexity. Since X is uniformly convex, δ is a strictly increasing continuous function with $\delta(2)=1$ and so the definition of $\alpha(x)$ has a sense. Moreover $\alpha(x)>1$.
He improved the above fixed point theorem of Bynum as follows.

Theorem 2. : If X is uniformly convex and

$$d(X,Y) < \alpha(X).\, WCS(X)$$

then Y has the fixed point property.

Prus noticed that the constant $\alpha(X)$ is not large enough and obtained the value of $\alpha(1_2)$ as 1.0138. He defined a function $f(t)$ by

$$f(t) = \left(\frac{1+t^p}{t^p + (1-t)^p}\right)^{1/p}$$

for $t \in [0,1]$ and proved the following theorem.

Theorem 3. : If $d(1_p, Y) < C_p$ where C_p is the maximum value of $f(t)$ attained in $[0,1]$ for some $1 < p < \infty$, then Y has the fixed point property.

By calculation it follows that

$$C_p > f(1) = 2^{1/p} = WCS(1_p).$$

UNIFORMLY NORMAL STRUCTURE

Gillespie and Williams [11] attempted to obtain the fixed point theorem of Kirk (Theorem 1) by withdrawing the condition of reflexivity of the space and imposing a stronger condition on normal structure, called uniformly normal structure. The Banach space X has *uniformly normal structure* if there exists a number h, $0 < h < 1$ with the property that if C is a closed convex bounded subset of X, then there exists x in C such that

$$\sup \{ \| x-y \| : y \in C\} \leq h.\text{diam}(C).$$

Uniformly normal structure clearly implies normal structure but the converse is not true. Consider the Banach space C, the space of all convergent sequences of real numbers with

$$\| x_i \|_C = \sup \{ | x_i | : i \in N\}.$$

Define a linear, one-one and continuous function g from C to l_2 by

$$g(\{ x_i \})= \{x_i/2^i \}.$$

Now renorm C, for $\{x_i\}$ in C as

$$||| \{x_i\} ||| = [\|\{x_i\}\|_C^2 + \| g (\{x_i\})\|_{l_2}^2]^{\frac{1}{2}} .$$

In [25] it is shown that this space has normal structure while in [11] it is obtained that this space, i.e. $< C, |||\,.\,||| >$ does not have uniform normal structure.

Bynum [7] showed that a uniformly convex Banach space has uniformly normal structure but the converse is not true. Renorm the space l_2 by

$$\|\{x_i\} \|_1 = \max \{|x_1|, (\sum_{i=2}^{\infty} |x_i|^2)^{\frac{1}{2}}\} .$$

Then $(l_2, \| . \|_1)$ is not uniformly convex but has uniformly normal structure.

Gillespie and Williams [11] proved that any nonexpansive selp-map of a closed convex bounded subset of a Banach space with a uniformly normal structure has a fixed point and proved the same result for Kannan-type maps. However, the following theorem of Bae [3] showed that the attempt made by them to remove the reflexivity in Theorem *1* becomes abortive.

Theorem 4. : Any Banach space with a uniformly normal structure is reflexive.

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V. M. SHAH - MAN AND MATHEMATICIAN*

N. V. Patel and J. R. Patadia**

Our first association with Professor V. M. Shah was that of a student-teacher relation and it is with great humility and delight that we are penning this write-up. Firstly, we were his students during our M.Sc. studies, thenafter his research students and then as his colleagues in the Department of Mathematics at the Maharaja Sayajirao University of Baroda (M.S. University of Baroda).

During our long association with him, we have seen him as a teacher, as a researcher, as an administrator, as a counsellor to his students and colleagues, a guardian lending helping hand, at times even in personal problems of his students and colleagues, as an organizer and even as a social worker thriving to spread scientific temperament, awareness and attitude in the society.

He truly believes that learning and teaching is a lifelong process and hence even long after his retirement, Professor Shah is academically quite active.

He has a long versatile experience of teaching not only to under graduate and post graduate students at the Faculty of Science, but also at the Faculty of Technology and Engineering of the University.

As a supervisor, we have found him, apart from being mathematically very sound, quite open and frank. Working with him is a pleasure; one always feels encouraged, free and tensionless.

Very considerate, liberal and loving as he is, during the moments of depression of a research student, he is a best counsellor and a friend. He is innovative, open to new ideas, and ready to accept the challenge.

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He was born in a noble family, hailing from a small village "Sarar" in Baroda district of the erstwhile State of Baroda. He had his Primary education in the same village, and his Secondary and University education in Baroda.

With a brilliant academic career at the M.S. University, he joined (July 1952) the Department of Mathematics, at the M.S. University as an assistant lecturer. Prior to his joining the Department, he was a merit scholar and the Fellow of the Faculty of Science of the University.

He worked under the supervision of Professor U. N. Singh for his Ph.D. degree.

Our joint research work has found place in some of the Journals of International repute e.g. the Journal of the American Mathematical Society, the Journal of the London Mathematical Society, the Journal of Mathematical Analysis and Applications (U.S.A.) etc.

As a Head of the Department of Mathematics, Faculty of Science, M.S. University, he lead the department for a long time. He was a very able administrator. Seeing the rapid industrial growth of Baroda, he could foresee the need of introducing certain courses at the graduate and post graduate level so that students get ample opportunities for jobs, and was the first in the whole state of Gujarat to introduce courses like Linear Programming, Operations Research, Numerical Analysis and Computer Programming. This was done as early as in 1975. Computer Programming was also introduced at the post-graduate level. He knew the potential and ability of each of his colleagues and was always very careful to keep the right person for the right work for the optimum output-at times taking academically and otherwise bold decisions.

He was instrumented in establishing a nice Computer Laboratory in the Department for which he rightly commands all the credit.

He also commands credit for instituting the degree of M. Phil in the Faculty of Science during his tenure as Dean of the Faculty of Science of the University.

Committed as he is to the cause of education at all the levels, when new mathematics was introduced at school level in seventies in Gujarat State, he took initiative in several directions to offer his own contribution for the educational cause and service to the society at large.

He was the Founder Editor (1972) of a monthly "Ganit Darshan" published with a view to bring home the concepts of new mathematics to school teachers.

At the State level, his contribution to the progress and services of the "Gujarat Ganit Mandal" in nurturing and promoting mathematical activities has been quite significant. He also served the Mandal as its Secretary for four years and later as its President (1979).

He was the Director of the University Computer Centre (since 1983 till his retirement in October 1989). Simultaneously, he was also the Dean of the Faculty of Science (since May 1985 till his retirement).

He wrote an excellent introductory book "Computer Darshan" in Gujarati language for the benefit of students and a common man of the society. Written in a lucid and popular Gujarati language for creating Computer awareness, the book has acclaimed wide popularity.

He is a Founder Secretary of a voluntary organization "Gujarati Vignan Parishad" engaged in Science Popularization activities.

He is also an invitee on the Governing Council of "Community Science Centre" in Baroda. The centre is engaged in organizing programmes for pupularization of Science among masses. He actively participates in the programmes such as seminars, Science fairs, celebration of Science day etc. by the centre.

He is a Fellow of the National Academy of Science. He is also a Founder Fellow of the Gujarat Science Academy.

He was a member of the UGC sponsored Curriculum Development Centre for Mathematics, set up at the Ramanujan Institute of Madras University.

Professor Shah has delivered series of postgraduate level lectures in several universities. He has also delivered series of lectures during several seminars and refresher courses organized at several university departments.

He was the President of the Mathematics Section (1989-90) of the Indian Science Congress Association, and presided over the Section during the Seventy Seventh Annual Conference of ISCA held under the auspices of the University of Cochin, in February 1990.

He was the President of the Indian Mathematical Society (1990-1991) and Presided over the Fifty Sixth Annual Conference of the Society held under the auspices of the South Gujarat University, Surat in December 1990.

Earlier, served the Society as its Treasurer (1986-1990) and successfully undertook the challenging task of setting this office in order. He also served the Society as Editor of its periodical "Mathematics Student" (1992-1996).

Presently, he (since April 1996) is the Editor of the Journal of the Indian Mathematical Society (JIMS), published by the Society.

He, with the cooperation of like minded people, frequently conducts a one-day or a two-days workshop for Primary school teachers and parents with a view to inspire and enrich them with regard to their Science and Mathematics curriculum, as also to develop scientific tmeper in them.

In fact a "Primary Teachers Forum" is established in Baroda for this purpose, with Professor Shah as its President.

He always entertained a feeling that our children should be moulded for logical reasoning, and for this purpose efforts should be made to develop a scientific kind of thinking habit among them. He strongly believes that such trained children, in return, would not only become better scientists, but also citizens with scientific temper.

With profound confidence in his belief that Mathematics is the only tool available and handy at the primary stage of our education system for achieving this high goal, he has undertaken a task of preparing a series of five books, coined agewise, for the children of age group 8 to 12.

The series is titled "Majedar Ganit" (Enjoyable Mathematics). Among several other things, the text of the series include a word to the parents, mathematical games, puzzles, fun, some historical tips, some touches on mathematics in ancient India including Vedic mathematics, some-poem-like material etc.

On his retirement, his admirers and well wishers created a "Felicitation Fund" by constituting a committee for the purpose. Admirers and well wishers willingly contributed to the fund. Thenafter, the committee distributed the amoung for various purposes e.g. Institution of Professor V.M. Shah gold medal at the M.S. University, prizes for the Post Graduate Mathematics Students of the University, V. M. Shah prize for the best research paper presented (in Analysis) at the Annual Conference of the IMS, an endownment fund to the Gujarat Ganit Mandal; and creation of a public trust : "Professor (Dr.) V. M. Shah Charitable Trust" at Baroda, for carrying out various socially useful activities of academic nature.

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H. C. KHARE AND MATHEMATICS EDUCATION

S. L. Singh*

Professor Harish Chandra Khare was born on October 5, 1927 at Fatehpur (U.P.). After completing his school and college education at the Government High School, Fatehpur and B.N.S.D. College, Kanpur, he joined the University of Allahabad in 1944 and took his B.Sc. and M.Sc. (Mathematics) degrees in 1946 and 1948 respectively. After a short stay as a Lecture in the Ewing Christian College, Allahabad, in 1948-49, he joined the Department of Mathematics, University of Allahabad, in 1949 as Lecturer. The same year he was selected for the Superior Forest Service but declined the appointment and chose to stay in the University. He went to McGill University, Canada, in January 1956 on a fellowship and obtained his Ph. D. degree from this University in 1960. During his stay at McGill University, he served as a Visiting Assistant professor in the Department of Mathematics. He, also started the Computer Centre and lectured on Numerical Analysis and Computer Programming at McGill University. He returned to the University of Allahabad in 1960. He was appointed Reader in 1964 and Professor of Mathematics in 1976. He was the Dean, Faculty of Science, University of Allahabad (1985-87). He retired from the University in June 1998.

He has to his credit 50 research papers published in reputed journals in India and abroad. His work on Position annihilation was the first of its kind. His publications are in the fields of Theoretical Physics, Magnetohydrodynamics, Plasma Dynamics and Fluid Mechanics. He has lectured at various Universities in India and abroad. He was U.G.C. National lecturer, 1985. He continues to guide research for the Doctor's degree at the University of Allahabad.

He was invited to and participated at the first International Conference on Position Annihilation at Wayne State University, Detroit, U.S.A. in 1965. He on invitation participated in the Seminar on Higher Applied Mathematics at the University of Montreal, Montreal, Canada in 1975.

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He is a Fellow of the National Academy of Sciences, India. He has held various offices in the National Academy of Sciences, India. He had been its General Secretary (1981-84) and (1989-1993), Chief-Section A (1975-80 and since 1985), Managing Editor (1981-84). He was vice-president, National Academy of Sciences, India (1985-1986). He was Foreign Secretary, National Academy of Sciences, India (1986-88).

He was the President Mathematics Section, Indian Science Congress Association, January 1985. He was for several years member of the Council of Indian Science Congress Association. He was the President of Indian Mathematical Society 1986-87 and member of its Executive Committee. He is the General Secretary Indian Mathematical Society.

Professor Khare has been keenly interested in the development of education in the country. He was member of U. G. C. Mathematics Panel and also its Convener. Professor Khare was appointed member, University Grants Commission by the Government of India in 1984 and served till 1987. He was associated with the National Commission on Teachers of the Government of India as a member of its working group.

He was appointed by Government of India a member of the Indo-US Sub-Commission on Education and Culture. He had been a member of the Executive Councils of the Banaras Hindu University, Jawaharlal Nehru University and of the University of Allahabad. He was a member of the Establishment Committee of the National Council of Educational Research and Training, New Delhi. He was a member of the first Executive Council of the newly established Pondichery University. He was a member of the Society of the Indian Institute of Advanced Study, Simla. He was a member of the Working Group on Higher Education Eighth Five Year Plan.

He was member of a University Grants Commission delegation to USSR. The delegation visited USSR in September 1986 to finalise bilateral agreements in the field of higher education. He went as U. G. C. nominee to attend an International Symposium on Informatics and the Teaching of Mathematics in developing countries at Monastir, Tunisia, in February 1986. He attended, on behalf of the U. G. C., the South Asian Region Workshop of Commonwealth Association of Science, Technology and Mathematics Educators at Singapore, in April 1986.

Professor Khare has been associated with many bodies and committees relating to education in general and mathematics in particular at the National level.

He has been connected with a large number of Universities in various capacities. He took an active part in the restructuring of courses in Mathematics at several Universities. During his service at the University of Allahabad he held many offices.

Professor Khare was invited to the Sixth International Congress on Mathematics Education held at Budapest in July/August 1988 to give a talk on Mathematics and other subjects. He attended this Congress and also visited institutions of higher learning and research in Hungary.

He is now the Treasurer, National Academy of Sciences, India since January 1996. He is the Chief Editor (Physical Science). Proceedings National Academy of Science, India.

Professor Khare is at present Emeritus fellow, University Grants Commission. He is the Coordinator in Mathematics for the preparation of undergraduate Mathematics tele Courses.

Professor Khare was awarded N. R. Sen Memorial Lecutre of Calcutta Mathematical Society 1989 Platinum Jubilee Lecture of Indian Science Congress Association 1991, Distinguished Service award by Mathematical association of India April 1989, P.L. Bhatnagar Memorial Lecture Award. Indian Mathematical Society 1991.

He was member of the Executive of the All India Council of Technical Education New Delhi (1994-1997). He was on the governing body of the Indian Institute of Tropical Meteorology Pune.

He has served on the Committees to prepare panel of names for Vice-Chancellor of various Universities in the country. He declined the Vice-chancellorship of H.N. Bahuguna Garhwal University.

Chairman National Workshop in Vedic Mathematics organised by Rashtriya Ved Vidya Pratisthan, held at University of Rajasthan, Jaipur 25-28 March, 1988.

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ASHOK KUMAR AGARWAL

Virendra Arora*

Professor A. K. Agarwal was born on August 18, 1949. He passed B.Sc. (1969) and M. Sc. (Math., 1971) from Agra University. He is a Ph.D. (1981) from I.I.T., Delhi. He has visited Pennsylvania State University, Mc Master University, Arizona State University, Univ. of Illinois, Auburm University, Wisconsin University, George Washington University, University of California/Minnesota/Qubec/ Victoria etc. His main research field is Number theory and he has published about 50 papers in reputed journals.

The following is a brife write-up of his work by himself.

"I did my Ph.D. in 1981 from IIT, Delhi under the supervision of Prof. H. L. Manocha. The title of my thesis was "Generating relations and differential operator representations". I went to USA in 1984 on a National Scholarship for Higher Study Abroad, Ministry of Education & Culture, Government of India. There under the Supervision of Prof. George E. Andrews, I shifted to partition theory and collaborated with Professors George E. Andrews, David M. Bressoud, Gray L. Mullen (all from the Pennsylvania State University, USA) and Prof. M. V. Subbarao of University of Alberta, Canada on various partition problems mostly related to Rogers-Ramanujan type identities. In our work we not only used known combinatorial objects anti-hook differences and lattice paths but also introduced two new kind of combinatorial parameters called partitions with

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"*n+t* copies of n" and partitions with "*d*(*a*) copies of *a*". While we used partitions with "*n+t* copies of *n*" in obtaining numerous Rogers-Ramanujan type identities, our study of partition with "*d*(*a*) copies of *a*" solved a problem of polynomial algebra posed by R.A. Hultquist, G. L. Mullen and H. Niederreiter. Partitions with "*n* copies of *n*" also arise in the solution of the Hard-Hexagon Model of R. J. Baxter.

I have also collaborated with Prof. Willard Miller (Univ. of Minnesota, USA) and Prof. H. M. Srivastava (Univ. of Victoria, Canada) on some special functions problems.

I am continuing my work in partition theory. Very recently I found a bijection between the partitions with "*n* copies of *n*" on the one hand and plane partitions on the other. This enabled me to obtain Rogers-Ramanujan type identities for plane partitions which were not found before. Thus I was able to fill in this gap in the theory of plane partitions. Very recently I succeeded in obtaining Ramanujan type-congruences for partitions with "*n* copies of *n*". Ramanujan type-congruences for perfect partitions I obtained in 1991 jointly with prof. M. V. Subbarao (Canada)."

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GURUKULA KANGRI PHARMACY

R. K. Rawat*

The Gurukula Kangri Pharmacy is preparing ayurvedic medicines of best quality from last one century. Our products created their own place in the hearts of public due to best standard in quality and high ideality in preparation as mentioned in old Ayurved books. Now, we are one of the largest ayurvedic drugs producer in India.

We are preparing about 400 products. We are also developing new products related to various disease. The potency of our medicines is far better and it is continuously increasing by modern techniques to improve quality. The main branches of medicines, which we are preparing are :

1. Powders : About twenty six
2. Bhasm : About twenty
3. Ras & Vati : About hundred twenty three
4. Pak - Avaleh : About twelve
5. Asava - Arishta : About twenty four
6. Medicines & Cosmetic oils : About fourteen
7. Ghrits : About six

And many new products are under consideration for preparation and testing.From above mentioned products the important products, which have leading sell are Bhimseni Surma, Chyavanprash, Payokil, Shilajeet, Gurukula Chai & Gold Preparations like Vrihat Vat Chintamani, Basant Kusumakar Ras & Yogendra Ras.

Since the unit is situated in the foothills of Himalaya so we are much capable to arrange fresh medicinal plants in comparison to other ayurvedic drugs producing units. Since we have plenty of land with us, so we are not only purchasing the fresh medicinal plants but also grow the required plants time to time. The identification of purchased medicinal plants is done by the expert ayurvedic doctors, and only after their approval, the plants or medicinal material is used for preparation. This factor maintained the quality of products in one way.

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The another way to improve quality is the methods of prepration of drugs. We are somewhat different with other ayurvedic drugs producing units because we are not compromising with the old methods of preparation of drugs given in standard books of Ayurveda. For this purpose, the strict supervision is done by the Ayurvedic doctors.

The washing of crude material, crushing and grinding of materials or products is done by the help of machines to make work more fast and for other preparative methods, we are strict to Ayurvedic book standard.

Our products specially Asavs and Arishtas are prepared in three to four months by self generating alcohol and not by mixing rectified spirit etc. so their potency towards diseases are very high.

Hence, from all points, we produly can say that due to quality, our products are more potent.

The packaging of medicines is done in different sizes of sealed packs. The packaging label has been printed by the weight of medicine, cost, year and batch number. Before purchasing medicine, it should be ensured that whether the seal is properly intact or not.

The drugs are sold through various agencies all over India.

WE EARN FOR CHARITY : The profit earned the pharmacy is used for charity. The profit is donated for the educatin of students at Gurukula Kangri Vidyalaya (Boys), Hardwar and Kanya Gurukul (Girls), Dehradun.

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VEDIC MATHEMATICS AND GKV

Virendra Arora*

The Gurukula Kangri (GKV) was found in 1902 by Svami Shradhananda Ji Maharaj (1856-1926). Svami Ji had a vision of having interaction between Vedic and modern sciences and this vision was of vital importance, especially because the country was in the hangover of European education and science.

The Vishwavidyalaya was having undergraduate department of mathematics in the College of Science and postgraduate department of mathematics in Ved and Arts College.

Principal S. L. Singh joined this Vishwavidyalaya as Professor of Mathematics in April, 1985, and subsequently the two department were merged in July 1985. He was instrumental in initiating Ph.D. programme (Mathematics) in the College of Science. Other science departments initiated Ph.D. programmes immediately thereafter.

Although Prof. Singh's main research field belongs to Operator Theory & Fixed Point Theory, his first love has always been "Lilavati". In GKV, he seems to be the first to start doctoral level research activities in Vedic mathematics and, of course, in the history of mathematics.

The first Ph.D. enrolment of this Vishwavidyalaya in Ancient Indian Mathematics was done under his supervision at the close of 80's.

Indeed, this was a good beginning for research activities in the science college. Now it is a matter of great pleasure that almost every science department is having a few master courses or doctoral level research work in Vedic sciences.

Our Mathematics Department has already organized two national level symposia on Vedic mathematics and Vedic geometry in 1989 and 1995 under the leadership of Prof. Singh.

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Functional Analysis, Topology, Operator/Optimization Theory, problems related with Computer Applications and Vedic mathematics are the main areas of research activities in mathematical sciences.

The following is a brief detail of Ph.D. theses awarded by the Vishwavidyalaya in this area :

Name	Title & Year of Theses		Supervisor
Ramesh Chand	Study of Siddhänta S'iromaṇi	(1990)	S. L. Singh
Vinod Mishra	Vedic geometry & its applications	(1997)	S. L. Singh
N. B. Singh	Indeterminate analysis & its applications	(1997)	S. L. Singh
Vivek Goel	A study of gaṇitatilaka	(1997)	Virendra Arora

True to his love for ongoing research activities in the tradition of ancient Indian Science, Prof. S. L. Singh has again initiated the present symposium on Vedic Mathematics during this conference which is compatible with Gurukula tradition. As evident from *Abstracts of Paper/Talks (Section N)*, it is a matter of great satisfaction that now other dipartments of the GKV have started taking interest in Vedic Mathematics and related areas. I take this opportunity to express categorically that in particular we and the GKV shall remain indebeted to the path shown by Prof. S. L. Singh.

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GURUKULA KANGRI VISHWAVIDYALAYA AND SCIENCE EDUCATION

Vinod Kumar*

In the middle of the eighteenth century, the kind of educational institutions available in India and institutions which were being established by the British Government and Christian Missionaries were not suitable to cater the educational requirements of Indian Society. The knowledge of modern science subjects could not get appropriate place in the old styled *Pathshalas, Madrasas and Maqtabas.* On the other hand, newly established schools were not having any place for Indian culture. Therefore, there was great need for an educational system, which could have appropriate place for the knowledge of modern subjects, and, at the same time, could generate faith among the students towards Indian culture and heritage. Another gigantic task before educationists was to overcome some superstitions like "*Strishudro nadhiyatan*" and to spread education among all the groups of the society.

Under these circumstances, by establishing Gurukulas, where the students from all walks of the society could equally participate, the Arya Samaj started a new era in the field of education. In these Gurukulas proper arrangements were made to study Snskrit Language, Ancient Literature and Vedic Literature along with the modern science subjects. Following the instructions laid down by Swami Dayanand Saraswati Ji , separate institutions were established for boys and girls. Gurukula Kangri Vishwavidyalaya, established by Swami Sharadhanand Ji in 1902 to give real shape to the dreams of his guru Swami Dayanand Saraswati Ji, proved to be a mile stone among such institutions .

The sole aim was to evolve an alternate system of education, to that British percepation based on English medium, to provide higher education of Vedic Literature, Indian Philosophy, Indian Culture and Literature along with modern subjects through establishing close contacts among the students and the teacher under *'Guru–Shishya'* progeny without distinguishing them on the basis of caste and creed. It helps in character building of the students through their moral and physical development.

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Manifold developments have taken place in the Vishwavidyalaya during the last 98 years. At present there exist four Colleges and six Faculties in the Vishwavidyalaya as listed below :

1. Ved and Arts college.
    - Faculty of Oriental studies
    - Faculty of Humanities
    - Faculty of Management studies

1. Science Collage.
    - Faculty of Science
    - Faculty of Life Science
    - Faculty of Technology

1. Kanya Gurukula Mahavidyalaya, Dehradun

2. Kanya Gurukula Mahavidyalaya, Haridwar.

The Science College :

The education of science subjects initially got started in this Vishwavisdyalaya in 1910. There were mainly two sactions *'Gurukula Vidyalaya'* and *'Gurukula Mahavidyalaya'*. In Gurukula Mahavidyalaya appropriate arrangements were made to provide education of modern science subjects like Chemistry, Physics, Botany and Mathematics through Hindi medium too.

At the same time Chemistry and Physics were introduced in the curriculum of Gurukula Vidyalaya and the medium of teaching, even of science subjects, was Hindi. During this period school level Hindi text books for Chemistry and Physics were written by Master Goverdhan. The efforts were also made to write graduate level textbooks of science subjects in Hindi. "Vanaspati shastra " and "Hindi Chemistry" by Prof. M.C. Sinha and 'Vikasvad' by Prof. V.G. Sathe were the first outcome of these efforts.

Although the foundation of the Science College was laid down in 1921 with the beginning of Ved College with the introduction of science subjects as a part of study, yet in real sense , the work on the development of a separate Science College started with the announcement of government grant of rupees one lakh by Dr. Rajendra Prasad (the first President of India) during his convocation address in 1950. The construction work of the building was immediately started . The Science College building was inaugurated by the then Prime Minister Pandit Jawaharlal Nehru on Aug.1,1958. Professor F. C. Trehan was appointed as the founder Principal of the college. Thereafter, the development process was kept on by Principal S. C. Tyagi and Principal Vijaya Shankar. All the Vice-Chancellors and principals have made tremendous efforts for the development of this college. The tenures of the present Vice-Chancellor Prof. Dharam Pal and the present Principal Prof. S. L. Singh have been a golden era in the development history of the Science College. The college is expanding its dimensions in all aspects, be it the construction of buildings, introduction of new courses or completion of research projects funded by various government agencies.

M.Sc, (Maths) and M.Sc. (Microbiology) courses were started in 1964 and 1984 respectively. M.Sc. (Physics) and M.Sc.(Chemistry) were intruduced in 1992. The introduction of such courses at PG level was really a great achievement. But the new technologies were yet to come up. It is in this sequence that Master of Computer Application (M.C.A.) was approved by the U.G.C. in 1993 and M.Sc. Environmental Science in 1994. At present the Science College is taking a leap forward in technological developments as well as in higher education in science. Vedic touch is the salient feature of all these courses.

Faculty & Departmental Profile

I. Faculty of Science

1. Department of Mathematics and Statistics

Courses offered	:	B.Sc & M.Sc.(Mathematics), (Special emphasis on Industrial Mathematics) Ph.D.
Research Areas	:	Functional Analysis, Topology, Vedic Mathematics, History of Mathematics, Queuing Theory, and Computer Communication Networks.
Faculty	:	S. L. Singh (Prof. & Dean/Principal), V. Arora (Reader) V. K. Sharma (Reader & Head), M. P. Singh (Reader), P. Pradhan (Lecturer).

2. Department of Chemistry

Courses offered : B.Sc. & M.Sc. (Chemistry) (Specialization in Commercial Methods of Chemical Analysis) Ph.D.

Research Areas : Bio-Inorganic Chemistry, Macrocyclic Chemistry, Electrochemistry, Plant-Medicines, Higher Valent Silver Reaction Mechanism, Falklore and Traditional Medicines, Complex Compounds.

Faculty :
R. K. Paliwal (Reader), A. K. Indrayan (Reader)
Kaushal Kumar (Reader), R. D. Kaushik (Reader)
Randhir Singh (Reader & Head), Srikrishan (Lecturer)
Professor's post is likely to be fulfilled soon.

3. Department of Physics

Courses Offered : B.Sc., M.Sc.(Physics) (Specialization in Instrumentation, Design & Maintenance), Ph.D.

Research Areas : Amorphous semiconductors and new materials, Atmospheric physics, Electronics

Faculty :
H. C. Grover (Reader), Rajender Kumar (Reader)
Parmanand Prakash Pathak (Reader & Head)

II. Faculty of Life Science

1. Department of Botany & Microbiology

Courses Offered : B.Sc., M.Sc.(Microbiology), Ph.D.

Research Areas : Microbiology, Micropropagation, Enhancement of therapeutic spectrum of indigenous drug and drug plants, Bio-Management of wastewater, Mvcology, Plant Pathology.

Faculty :
D. K. Maheshwari (Prof. & Dean),
P. Kaushik (Reader & Head),
G.P. Gupta (Sr. Lecturer),
Ramesh Chandra Dubey (Reader), Navneet (Lecturer)

2. Department of Zoology & Environmental Science

Courses Offered : B. Sc., MSc.(Environment Science), Ph.D.

Research Areas : Water pollution, Ecology, Fishery

III. Faculty of Technology

1. Department of Computer Science

Courses Offered : B.Sc., Master of Computer Applications (MCA), Ph.D.

Research Areas : Distributed Computing, Interconnected Networks, Performance Evaluation, Reliability Engineering.

Faculty : V. Kumar (Prof. & Dean), K. J. Bhatia (Sr. Lecturer)

2. Computer Centre :

Computer centre is equipped with state-of- the-art software and hardware facilities. It has a LAN with UNIX, WINDOWS NT and Novell servers with 40 nodes. All kinds of printer (line, DMP, laser, inkjet) & scanners (B/W, Color) are available in the centre. Internet and E-mail facility are also available.

Process of Admission :

Admission for B.Sc. level in various groups is made on the basis of the marks obtained by the candidate in the subjects required for that particular group. For admission to P.G. courses, a national level entrance exam is conducted at various centres of the country and the admission is granted in the individual P.G. courses.

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Abstracts
of
Papers/Talks

INDIAN MATHEMATICAL SOCIETY

64th Annual Conference

and

Symposia

on

Traditions of Vedic Mathematics & Applications

and

Fixed Point Theory in Nonlinear Analysis & Applications

December 19-22, 1998

Department of Mathematics and Statistics
Science College

GURUKULA KANGRI VISHWAVIDYALAYA
HARDWAR 249 404

Abstracts are classified into sections according to the following IMS code:

(A) Combinatorics, Graph Theory and Discrete Mathematics
(B) Algebra, Number Theory and Lattice Theory
(C) Real and Complex Analysis (including Special Functions, Summability and Transforms)
(D) Functional Analysis
(E) Differential Equations, Integral Equations and Functional Equations
(F) Geometry
(G) Topology
(H) Measure Theory, Probability Theory and Stochastic Processes, and Information Theory
(I) Numerical Analysis, Approximation Theory and Computer Science
(J) Operations Research
(K) Solid Mechanics, Fluid Mechanics, Geophysics and Relativity
(L) Electromagnetic Theory, Magneto-hydrodynamics, Astronomy and Astrophysics
(M) Bio-Mathematics
(N) History and Teaching of Mathematics

A:1

STUDY OF THE SOCIO-ECONOMIC PROBLEM USING COMBINED *FCM*

W. B. Vasantha and S. Uma
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Here we study the combined effect of the fuzzy cognitive maps for a socio-economic problem. By this method we obtain the combined *FCM* which cannot be done using usual logic. We arrive at the conclusion that poverty causes crime by using combined *FCM*.

A:2

SPLIT BONDAGE NUMBERS OF A GRAPH

V. R. Kulli, B. Janakiram and Radha R. Iyer
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Let $G = (V, E)$ be a graph. The negative split bondage number $b_{\bar{S}}(G)$ of G is the minimum cordinality among all sets of edges $X \subset E$ such that $\gamma_S(G\text{-}S) < \gamma_S(G)$ and the split bondage number $b_{\bar{S}}(G)$ of G is the minimum cardinality among all sets of edges $X \subset E$ such that $\gamma_S(G\text{-}X) > \gamma_S(G)$, where $\gamma_S\ (G)$ is the split domination number of G. In this paper, we initiate a study of these two new parameters.

A:3

CONSTRUCTION OF *q*-ANALYTIC FUNCTIONS

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In all the discrete analytic function theories, discrete analytic continuation is the standard process of defining a discrete analytic function from a domain where only the function is known fully. In q-analytic theory (Harman, 1972) such an operator is evolved from the limit points on the positive coordinate axes to continue to the whole discrete geometric space.

In this attempt, a function can be continued q-analitically from the basic distorted ray, basic semi raidan ray and its adjacent to the whole of discrete geometric space is evolved. In particular the formula to continue from the basic distorted ray to the first Traint is found. Discussing the anomalies and limitations in the existing theory, This continuation process is an improved version and near to perfection.

In discrete function theories, the modern trend is that a discrete analytic function means only a sequence of values of the function. The function is known fully in the whole space through analytic continuation. In this stream an attempt is made here to initiate a study of q-analytic functions.

A:4

FUZZY COGNITIVE MAPS REPRESENTING SOCIO-ECONOMIC MODEL

W. B. Vasantha Kandasamy

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Here we use fuzzy cognitive maps to find the hidden pattern of a socio-economic model with population, crime Economic condition poverty and unemployment as 5 concepts. In this model we study the 3 experts openion and obtain the hidden pattern of the model.

The main conclusion drawn from these experts view is when the population is on the increase we get a fixed point for crime which suggests that increase in population results in increase in crime. We obtain the inter relation between these 5 concepts.

B:1

OCTANIONS AND 8-DIMENSIONAL VECTORS

M. S. Saroa
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The field of complex numbers has been extended to Octanions (8-dimensional) and an identity, that is, the product of sums of eight squares is equal to the sum of eight squares is oftained. Similar result is oftained through the extesion of vectors to the system of eight dimensional vectors.

B:2

CONSTRUCTION OF FINITE GROUPOIDS USING INTEGERS

W. B. Vasantha and A. Govindarajan
Department of Mathematics, I.I.T. Madras and S.R.M. College,
Kattankulathur - 600 036

Here we introduce a new class of groupoids using integers 0, 1, 2, ..., n; (n 3) and study their properties. We further define left identity, right identity and identity in this new class of groupoids. A groupoid defined in this class depends on the pair (t, u); (t, u) ∈ {0, 1, ..., n}. Further it is proved a groupoid obtained using the pair (t, u) is different from the pair (u, t).

B:3

ON BOOLEAN ALGEBRAS

V. R. Chandran
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Prof. M. Rajagopalan (Memphis State Univ.) (Personal Communication) has asked for the existence of an atomless Boolean Algebra. In this paper, we Construct such an example using Lattiee theory and Ring theory. We also device an interesting cardlary *vig*^{1} There exists no atomless Boolean lattice with dual atoms !

B:4

GROUP MATRICES

W. B. Vasantha and S. R. Kannan

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Here we introduce a new algebraic structure which does not form a ring or a field or a domain called Group matrices. These group matrices are obtained from the class S of subgroups of a group G. We obtain some characterization theorems and results about them.

Let G be a commutative group. Then $M(G)_{nxn}$ is a semiring in which $A+B = E$ if and only if $A = E$ and $B = E$. Further in this case $M(G)_{nxn}$ is a semi-ring with an additive identity. If G is non commutative group then $M(G)_{nxn}$ is not a semi ring. If $M'(G)_{nxn}$ denotes only the class of normal subgroups of the non commutative group G then $M'(G)_{nxn}$ is a semiring.

B:5

A GROUP HOMOMORPHISM WHICH IS NOT A RING HOMOMORPHISM

Rajiv K. Srivastava

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A ring homomorphism of one ring to another ring is a group homomorphism under their respective group structures. The converse, it is understood, does not hold, in general. Yet, in literature, there is a dearth of an example of a mapping from a ring to another ring which is a group homomorphism between the respective group structures but fails to be a ring homomorphism.

In this paper, we provide such an example. The said mapping possesses many more interesting algebraic, geometric and functional analytic properties which have been studied in this paper.

B:6

A LINEAR ORDER ON A SET OF MAXIMAL CHAINS OF A SEMILINEAR ORDER

Shabeena Ahmed

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A linearly ordered set or a chain is a partially ordered set in which any two elements are comparable with respect to the partial order. A semilinear order generalises the notion of a linear order and a *C*-relation is a ternary (3-place) relation defined on the set of maximal chains of a semilinear order. We construct a semilinear order on the plane and consider a dense set of maximal chains with the underlying *C*-relation. We then impose a linear order on the set of maximal chains and study the Jordan sets of the automorphism groups of the set of maximal chains. The Jordan sets of the maximal chains with the *C*-relation have already been studied by S.A. Adeleke and P.M. Neumann. In this paper we give a complete description of the Jordan sets of the maximal chains preserving both the *C*-relation and the linear order.

B:7

A NEW IDENTITY IN LATTICES

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Here we introduce a new identity in lattices and call those lattices which satisfy this identity to be a *N*-lattice, we prove every *N*-lattice is modular and super modular. Further we prove every distribute lattice is a *N*-lattice and not conversely.

B:8

SEMI-MODULES OVER SEMI-RINGS

W. B. Vasantha and N. Suresh Babu

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Here we define a new notion called semi-modules over semi-rings.

We mainly prove that if M is a strict S-semi-module then S is a strict semi-ring. However even if S is a strict semi-ring we prove every S-semi-module over S need not be a strict-semi-module.

Further we define quotient semi-module and obtain some interesting results about them. We finally define tensor product of two S-semi-modules and pose it as open problem to carry over to prove results on tensor product of S-semi-modules.

B:9

CONCATENATION OF CODES WITH RANK METRIC

W. B. Vasantha and R. Sujatha

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The algebraic coding theory is one of the branches of communication theory which plays a vital role in the transmission of messages. To have a reliable transmission of messages, we need codes which can both detect and correct errors. Concatenation of codes is an important method to obtain lengthy codes with good error correcting capability. Here we are the first one to introduce concatenated codes with rank metric. These codes not only transmit the information reliably but also help us in detecting and correcting errors.

B:10

HAMILTONIAN GROUP RINGS

W. B. Vasantha Kandasamy and V. Vaithiyanathan

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Here we obtain conditions under which the group rings are Hamiltonian or otherwise. Jin Xiang Xin was the first one to define generalized Hamiltonian Rings. He calls a ring R to be a generalized Hamiltonian ring if every non zero subring of R includes a non zero two sided ideal.

We prove the group ring Z_2G of a group G over Z_2 which contains an element of order two is not Hamiltonian. Thus Z_2S_n is not an Hamiltonian group ring for every S_n for all n. In case of torsion free groups G the question whether the group ring RG is Hamiltonian remains unsolved.

B:11

SUPER MODULARITY OF SUBLOOPS IN LOOPS

W. B. Vasantha Kandasamy and S. V. Singh

Department of Mathematics, I.I.T., Chennai 600 036

Here we study when does subloops of a loop form a supermodular lattice. We obtain a characterization theorem for the subloops of a finite loop L to be a supermodular lattice.

The collection S of all subloops of a finite loop L is a supermodular lattice if and only if S does not contain three distinct subloops such that one and only one pair is comparable and S does not contain a set of four distinct subloops S_1, S_2, S_3, S_4, such that $S_1 \cup S_2 = S_1 \cup S_3 = S_1 \cup S_4 > S_1$ and $S_1 > S_2 \cap S_3$, $S_2 \cap S_4$ and $S_3 \cap S_4$.

B:12

FORMAL INTEGRATION IN COMPOSITION NEAR- RING

S. Nabachandra Singh

Department of Mathematics, Manipur College, Imphal 795 008

The purpose of this paper is to introduce a formal integration operator for composition near-ring and to study about its recurrance and about its properties on some well-known composition near-rings.

B:13

A NOTE ON UNITIFICATION PROBLEM

B. N. Waphare and S. S. More

Department of Mathematics, University of Pune

S. K. Berberian raised the open problem "Can every weakly Rickart*-ring be embedded in a Rickart *-ring ? With preservation of *RP's*?" N. K. Thakare and B. N. Waphare gave a partial solution a weakly Rickart-ring *A* can be embedded in a Rickart *- ring provided there exists a ring *K* such that

(i) K is an integral domain with involution,

(ii) A is a *-algebra over K,

(iii) for any $\lambda \in K - \{0\}$ there exists a projection $e_\lambda \in A$, such that $x \in A$, $\lambda_x = 0 \Rightarrow LP(x) \leq e_\lambda$.

In this paper a more general partial solution of the problem is obtained. More precisely in the above partial solution the integral domainness of K is replaced by arbitrary commutative ring with identity.

B:14

ON STONE POSETS AND STRONGLY PSEUDOCOMPLEMENTED POSETS

M. M. Pawar and B. N. Waphare

Department of Mathematics, University of Pune

In this paper some of the results in Halas [2] are strengthened. The concept of strongly pseudocomplemented poset is introduced and obtained its characterization and established its connection with Stone poset.

B:15

A CRITERION FOR DIVISIBILITY OF NUMBERS

K. K. Velukutty

Department of Mathematics, SRM Vidyalaya Coimbatore 641 020

There are well known and simple divisibility criterion for numbers like 3, 7, 9 and 11. A uniform rule is not available. An attempt is made here to establish a criterion for and to formulate a unified approach to divisibility. The criterion reads as : N is divisible by r if and only if r is a factor of 10^1-t and the sum of the products of the syllables of N of order 1 and the corresponding multipliers of t is divisible by r.

In many cases this practical procedure is very simple and in the rest not at all tough. Various formulae are introduced in a tabular form for divisibility by smaller numbers; cases like those of 37, 58, 77, 91 and 143 are demonstrated to be simple.

A two man game is formulated between the offender and the defender such that the defender always wins; certain special cases are demonstrated. This note is important in two aspects : as a unified theoritical development for criterion for divisibility of numbers as well as a recreational game on numbers.

B:16

PROOF OF FERMAT'S LAST THEOREM USING ITS ORIGINATOR'S METHOD

V. K. Gurtu

Laxminarayan Institute of Technology, Nagpur University, Nagpur 440 001

Andrew wiles (1995) has published 109-page proof of Fermat's Last Theorem (*FLT*) by identifying it with a structural conjecture in Number Theory involving elliptic curves. However, in the present exposition, we have proved *FLT* for a general odd prime by adopting techniques which were used in Euler's proof of *FLT* for Cube. During the process we have introduced only a minor change in terms of identifying one of the two equations obtained from the governing equation of *FLT* in a slightly different way. But, significantly, this minor change has simplified the proof to such an unimaginable extent that it becomes almost one with the tought process of the originator of *FLT*.

B:17

ON RADICALS OF NEAR-RINGS

Manoj Kumar Manoranjan

Department of Mathematics, B.N.M.V. College, Madhepura 852113

Here the study of radicals of finite near-rings is initiated. The result gives a description of hereditary radicals having hereditary semi-simple classes. Since any radical in the class of associative rings has a hereditary semi-simple class. Also in the variety of not necessarily associative rings hereditary semi-simple classes are quite rase. General radical theory of near-rings paid special attention to the hereditariness of semi-simple classes. Also finite near-rings apply the structure theory of near-rings with DCC on right N-sub groups.

B:18

A REMARK ON FERMAT'S THEOREM

Mangey Singh
R.S.S. Inter College Jhinzhana, Muzaffarnagar

C:1

AN EXTENSION OF GENERAL CLASS OF BILATERAL GENERATING FUNCTION OF MODIFIED JACOBI POLYNOMIALS

M. A. Hussain and Faizi Aftab
Department of Mathematics, V.K.S.U., Ara, (Bihar)
Department of Mathematics, Oriental College, Patna 800 006

In this paper, we extend the general theorem on bilateral generating function of modified Jacobi polynomials. Some applications have also been deduced.

C:2

DISTRIBUTIONAL ABELIAN THEOREMS FOR A GENERALIZED STIELTJES TRANSFORM

Anil Kumar Mahato and J. M. C. Joshi
Department of Mathematics, Ranchi University
Department of Mathematics, Kumaon University, Nainital

In this paper an extension of the generalized stieltjes transform

$$F\ (s) = \Gamma(\rho) s^{m\rho-1} \int_0^\infty \frac{f\ (t)}{(s^m+t^m)^\rho} dt \quad (\mathrm{m,p} > 0)$$

to generalized functions is provided. Some Abelian theorems for the distributional generalized Stieltjes transform are proved and application of the Abelian theorem is given.

C:3

ASYMPTOTIC REPRESENTATION OF WEYL FRACTIONAL INTEGRALS

P. K. Banerji, Deepali Sinha and Megumi Saigo
Department of Mathematics, JNV University, Jodhpur 342 005

Object of the present article is to obtain asymptotic representation

of Weyl fractional integral, based on a method, employed by Kilbas for the Riemann-Liouville fractional integral. We have obtained the asymptotic representation for the case $\mu_k \neq p\ (j + 1)$ and, further, have formulated the asymptotic representation for n- dimensional fractional integral of $|x| \to \infty$ for the case $\mu_k \neq n + 2j$.

C:4

AN ALTERNATIVE PROOF OF SZEGÖ TYPE RELATION FOR THE SET $\{K_n(\alpha, \beta, k\ ; x)\}$

C. B. Kshatriya and P. G. Andhare
Department of Mathematics, H.P.T./R.Y.K. College, Nashik
Department of Mathematics, RBNB College, Shrirampur

Madhekar and Thakare defined and studied biorthogonal polynomials $\{J_n(\alpha,\beta,k;x)\}$ and $K_n(\alpha,\beta,k;x)\}$, suggested by Jacobi polynomials. In present note we give an alternative proof of Szego type relation for $\{K_n(\alpha,\beta,k;x)\}$.

C:5

ON BAILEY'S FORMULA INVOLVING PRODUCT OF GENERALIZED HYPERGEOMETRIC SERIES

Arjun K. Rathi
Department of Mathematics, Dunger College, Bikaner 334 001

The aim of this paper is to obtain single series expressions for

$$F\ (-;\ \rho;\ x)\quad F\ (-;\ \rho + i;\ -x)$$

for i=1 and -1. For i=0, we have the well known result due to Bailey. The results are derived with the help of Generalized Kummer's theorem very recently obtained by Lavoie, Grondin and Rathie.

C:6

A NOTE ON KUMMER'S SECOND THEOREM

Arjun K. Rathi and Manish K. Sareen
Department of Mathematics, Dunger College,
Bikaner 334 001

The aim of this research note is to derive the well known Kummer's second theorem from a Bailey's formula involving product of generalized hypergeometric series.

C:7

A UNIFYING APPROACH TO A NEW INTEGRAL TRANSFORM

P. K. Banerjee and D. N. Vyas
Department of Mathematics, MLV Textile Institute,
Bhilwara 311001

The object of the present paper is to define a new integral transform which is quite general in character. Special cases in terms of knwon integral transform are also given. Its relation with fractional calculus has been reported and a table of integral transforms or certain elementary functions is also provided.

C:8

LOCALIZATION OF $|\tilde{N},p_n|_k$ SUMMABILITY OF FOURIER SERIES

Syed M. Mazhar
Department of Mathematics, Kuwait University, Safat 13060, Kuwait, P.O. Box 5969

The paper deals with the localization problem for absolute summability of Fourier series.

C:9

STIELTJES TRANSFORM OF A CLASS OF GENERALIZED FUNCTIONS

P. K. Banerjee, Kuldeep S. Gehlot and Megumi Saigo
Department of Mathematics, JNV University Jodhpur 342 001
Department of Applied Mathematics, Fukuoka University, Fukuoka, Japan 814 080

In this paper a generalization of stiltjes transform, involving Gauss hypergeometric function in the kernel, is extended to generalized functions. Two testing function spaces are constructed, properties of testing function space and its duals are studied. Transformable generalized function are defined and analyticity theorem is proved for the generalized integral transform.

C:10

VALUE DISTRIBUTION OF FOURTH PAINLEVE DIFFERENTIAL EQUATION

S. S. Bhoosnurmath and C. M. Hombali
Department of Mathematics, Karnatak University, Dharwad

In this paper we consider fourth painleve type differential equation

$$ff^{(4)} = f^{(3)} f' - \frac{f^{(2)})^2}{2} + 20f^6 + 96f^5 + 90f^4 + 80zf^3 + 60(z^2 - \alpha)f^2 + B$$

Where B is some constant. Applying Nevanlinna theory we study the value distribution properties of transcendental meromorphic solutions of such equations. For the transcendental mermorphic solutions f of the above equation, we make an interesting observation that the second fundamental theorem of Nevanlinna goes into asymptotic equality.

C:11

ON THE DOMAIN OF NORMALITY IN ITERATION THEORY

P. Bhattacharyya

Department of Mathematics, IIT, Chennai 600 036.

In this paper we prove some new results on the components of the FATOU SET $F(f)$ which is the set of normality (in the sense of Montel) of the sequence of natural iterates of a function of a complex variable z.

In particular we prove that if f is a finite map, then $F(f)$ cannot have more than two components.

C:12

ON CERTAIN q-SERIES IDENTITIES

T. B. Jagtap

Department of Mathematics, R.B.N.B. College, Shrirampur, Ahmednagar 413 709

This paper incorporates a q-series indentity along with its interesting particular cases.

C:13

ON CERTAIN q-SERIES AND CONTINUED FRACTIONS

Remy Y. Denis

Department of Mathematics, Gorakhpur University,
Gorakhpur 273 009

Continued fractions representation for certain Mock theta functions have been established. The paper concludes with a brief discussion of the importance of this type of representations.

C:14

SOME POLYNOMIALS ASSOCIATED WITH A GENERALIZED POLYNOMIAL SET $\{P_n^{\alpha,\gamma}(x,r,s,p,k,\lambda)\}$

P. G. Andhera and T. B. Jagtap
Department of Mathematics, R.B.N.B. College, Shrirampur 413709

Patil and Jagtap studied the polynomials $\{P_n^{\alpha,\gamma}(x,r,s,p,k,\lambda)\}$. Here we define and study some polynomials associated with generalized polynomial set.

C:15

AN ESTIMATE OF A PARTIAL SUM OF THE FOURIER SERIES

Prem Chandra and Aalok Verma
School of Studies in Mathematics, Vikram University, Ujjain 456 010

In this paper, an estimate of a partial sum of the Fourier series of $f \in L_{2\pi}$ at a point under more general conditions has been obtained which leads to some known estimates under weaker conditions imposed upon the generating functions of the Fourier series. An application of this estimate also yield an interesting criterion for the convergence of the Fourier series at a point.

C:16

SOME RESULTS ON MATRIX TRNASFORMS OF H-FUNCTION

Vilas D. Raghatate
Department of Mathematics, S.S.G.M. College of Engineering, Shegaon 444203

In this paper we have established some new results related to Matrix transforms of *H*-function the *H*-function is defined by Innayat Hussain in 1987 is more generalised form of Fox's *H-function*. Many author's have established results on matrix trnasforms of Meijer's *G-function* and also Fox's *H-function* and many other functions.

C:17

ON THE PARTIAL SUMS OF STARLIKE FUNCTIONS

Sukhjit Singh

Department of Mathematics, S.L.I.E.T., Longowal

Let $S_t^{(½)}$ denote the usual class of starlike functions of order ½. If $S_n(z,f)$ denote nth partial sum of f, then it is well knwon that Re $[f(z)/S_n(z,f)] > ½$, for all $f \in S_t^{(½)}$ and for all z in the unit disc $E = \{z / |z| < 1\}$.

In the present paper, among other things, we give two way generalization of this results and prove : If $f \in S_t^{(½)}$ then for all t, $0 \leq t \leq 1$ and for all $n \in N$, we have (i) $Re[tf(z)/S_n(tz, f)] > ½$, $z \in e$ and (ii) $Re[tf(z)/\sigma_n(tz, f)] > ½$, $z \in E$. Here $\sigma_n(z, f)$ denote the nth cesaro sum of order i, i=1,2,3.

C:18

SOME BILATERAL GENERATING RELATIONS FOR THE FAMILY OF GENERALIZED POLYNOMIAL SYSTEMS

G. N. Dave

Department of Mathematics, Govt. P.G. College, Kotdwar, Garhwal.

Earlier, Agarwal and Dave have defined a family of generalized polynomial systems $B_n^B\ \{(x_m),Y\}$, $B_n^C\ \{(x_m),Y\}$ and $B_n^D\ \{x_m),Y\}$ by means of generating relations involving the generalized hypergeometric functions p_q^F and Lauricella's functions F_B, F_C and F_D. We have already discussed their explict forms, generalized Lauricella forms and n^{th} differential formulae. In this paper several bilateral generating relations of these polynomials has established and their relations with classical bilateral generating relations discussed in detail.

C:19

ON A SEQUENCE OF FOURIER COEFFICIENTS

K. N. Singh

Department of Mathematics, S.G.R. (P.G.) College Dobhi Jaunpur 222 149

In the present paper we prove a theorem on $C_1 \cdot (N,p,q)$ summability of the sequence $\{nB_n(x)\}$ under very general condition.

The $C_1 \cdot (N,p,q) \ \# \ (N,p,q).\ C_1$ except when $\{p_n\}$ is defined as silverman.

If $q_n=1$ for all n, the (N,p,q) means reduces to the (N,p_n) means and if $p_n=1$ for all n, the (N,p,q) means reduces to the (N,q_n) means of the sequence $\{S_n\}$.

C:20

ON THE ILYEFF - SENDOV CONJECTURE

Prasanna Kumar N.

Department of Mathematics, Mangala Gangotri University 574 109

A conjecture due to Sendov, better known as Ilyeff-Sendov conjecture is stated as follows,

CONJECTURE : Let $P(z)$ be a polynomial of degree $n \geq 2$ with all its zeros in the unit disk $|z| \leq 1$. If a is any one of these zeros, then $P'(z)$ has atleast one zero in the disk $|z-a| \leq 1$.

This conjecture has been verified for all polynomials of degree n, $2 \leq n \leq 5$. In this note, I verify the velidity of the conjecture for polynomials of degree n with atmost five distinct zeros. I reached the solution from a different direction and a shortest route which improves all earlier results on this.

D:1

AN APPLICATION OF SCHAUDER'S FIXED-POINT THEOREM

B. S. Yadav

TU-67, Vishakha Enclave, Pitampura, Delhi

Schauder's Fixed-Point Theorem states that every convex compact subspace of a Banach space is a fixed-point space. By using a variant of this theorem, Lomonosov proved a theorem on the existence of non-trvial invariant subspaces of non-zero compact operators. We discuss in this *TALK* the present state of Lomonosov's theorem in the context of recent developments in the invariant subspace theory.

D:2

ON ADJOINTS AND SPECTRA OF SHIFT OPERATORS ON WEIGHTED SEQUENCE SPACES

Bhopinder Singh

Govt. College of Engineering and Technology, Jammu

For $a > 0$ and for $1 \leq p < \infty$, let $l^p(a)$ be the Banach space of all complex sequences $x = \{x_n\}_{n \in N}$ for which

$$\|x\| = \left(\sum_{n=1}^{\infty} a^n |x_n| p\right)^{1/p} < \infty.$$

In this paper, the adjoints and the various parts of spectra of the right shift and left shift operators on $l^p(a)$ have been calculated.

D:3

FIXED POINT THEOREMS FOR BANACH OPERATORS ON PRODUCT SPACES

A. B. Lohani

Department of Mathematics, Kumaon University, Nainital 263 002

In this paper we introduce the Banach operators on finite product of 2-metric spaces and obtain fixed point theorems for such operators.

D:4

ON CONTINUITY AT FIXED POINTS

R. P. Pant

Department of Mathematics, Kumaon University, Nainital 263 002

The aim of this paper is to provide an affirmative answer to the open question on the existence of a contractive condition which is strong enough to generate a fixed point but does not force the map to be continuous at the fixed point. We do this by using the notion of reciprocal continuity and obtain a common fixed point theorem which significantly improves the known fixed point theorems for four mappings.

D:5

COMMON FIXED POINT THEOREMS FOR R-WEAKLY COMMUTING MAPPINGS OF TYPE (A_f)

R. P. Pant and S. Padaliya

Department of Mathematics and Computer Science, Kumaon University, Nainital 263 002

In this paper is to study some consequences of recently introduced notion of R-weak commutativity of type (A_f) by Pathak *et al.* [10]. We obtain some fixed point theorems for compatible and non-compatible mappings. Also we have employed the notion of pointwise R-weak commutativity of type (A_f) is a necessary condition for the existence of a common fixed point of contracive type mappings.

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D:6

ON COMMON FIXED POINTS OF ASYMPTOTICALLY REGULAR MAPPINGS

Praveen Kumar Shrivastava, N. P. S. Bawa, Vivek V. Raich and Pankaj Singh
Govt. P. G. College, Balaghat, Govt. Science College, Rewa

Without considering the usual sequence of successive approximations (an idea developed by Rhoades and Sessa *et.al.*), the results of this paper are established in metric space (X, d) in order to show the existance of common fixed points under following contractive condition;

$$d(Sx, Ty) \leq a_1 d(Sx, Ax) + a_2 d(Tx, Ax) + a_3 d(Sy, Ay)$$

$$+ a_4 d(Ty, Ay) + a_5 d(Sx, Ay) + a_6 d(Tx, Ay)$$

$$+ a_7 d(Sy, Ax) + a_8 d(Ty, Ax) + a_9 d(Ax, Ay)$$

for all x, y in X.

The main result extends and generalizes essentially known results of Das and Naik, Fisher, Jungck, Rhoades and Sessa et. al., Singh and Tiwari and Several others.

D:7

COINCIDENCE POINTS FOR HYBRID CONTRACTION

Praveen Kumar Shrivastava, N. P. S. Bawa and Vivek V. Raich
Govt. P. G. College, Balaghat, Govt. College, Amarpatan, Satna

Using the idea developed by Singh et. al. sufficient conditions for the existance of coincidence points for a very general type of contractive condition involving two pairs of mappings. Where each pair consists of a single-valued and a multi-valued mappings (known as hybrid contractin) are obtained. Main result of the paper extends and generalizes the results of Naidu, Sastry, Naidu and Prasad.

D:8

ON CHARACTERIZATIONS OF FIXED POINT OF CONTINUOUS MAPPINGS

Renu Chugh and Sanjay Kumar

Department of Mathematics, M.D. University, Rohtak

Some results concerning necessary and sufficient conditions for the existence of fixed points of continuous mappings in metric spaces have been proved by Dien and Liu, Xu and Cho using compatibility. These results generalize, improve and unify the corresponding results of Chang, Fisher, Dien and Jungck. In this paper we obtain the same result but with a generalized contractive condition which gives in particular the result of Liu, Xu and Cho. Our results generalize, improve and unify corresponding results of Chang, Fisher and Jungck.

D:9

ON THE KNASTER - KURATOWSKI - MAZURKIEWICZ - SHAPLEY - FAN THEOREM

D. D. Sharma

Department of Mathematics and statistics, GKV Hardwar 249 404

For almost five decades *KKM theorem* have been riding a wave that has enveloped many areas of mathematics, social and natural sciences in its power, creativity and expanse. Traveling far beyond the traditional bound of mathematics and science to distant shores of popular culture, this wave have captured the attention and enthusiasm of a worldwide audience.

In this *talk* our aim is to present a survey of generalizations of *KKM* theorem and some of its applications in nonlinear analysis.

D:10

COMPLEX 2-BANACH ALGEBRAS

E. Keshava Reddy, S. Bhattacharya and S. N. Lal
Department of Mathematics, Banaras Hindu University, Varanasi

In this paper we have introduced the notion of complex 2-Banach algebras and among several other result it has been shown that if x is in a Complex 2-Banach algebra then the spectrum $\sigma(x)$ is a closed bounded non-empty subset of the Complex plane C.

D:11

COMMON FIXED POINTS OF SET-VALUED MAPPINGS

P. P. Murthy
Department of Mathematics, Arignar Anna Govt. Arts College, Karikal 609605

The purpose of the present paper is to study continuation of the author of this paper with Singh and Singh ([11]). These theorems in this paper are indeed extension of the Gregus [06], Fisher and Sessa [5], Diviccaro, Fisher and Sessa [02], Murthy, Cho and Fisher and many results of set-valued mappings.

D:12

THE COMETRIC PROJECTION AND FIXED POINTS

Geeta S. Rao
Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai 600 005

Let X be a normed linear space over the real numbers and let G be a nonempty subset of X. Let $x \in X \setminus \overline{G}$ and $g_0 \in G$. Then g_0 is an element of best coappromimation to x if $\|g_0 - g\| \leq \|x - g\|$ for every $g \in G$. If R_G is the map associating with x, the set of all elements of best coapproximation to x, then R_G is called the cometric projection. Some results concerning cometric projection and fixed points are presented. Releated maps are defined and discussed. Certain connected ideas are elaborated.

D:13

A NONLINEAR ERGODIC THEOREM FOR MULTIVALUED OPERATORS

S. L. Singh

Department of Mathematics and Statistics, GKV Hardwar 249 404

The first nonlinear ergodic theorem, due to J.-B. Baillon, says that sequences of iterates of a nonexpansive operator on a Hilbert space (with some condition) are almost convergent to a fixed point of the operator. The purpose of this talk is to offer a brief report on nonlinear ergodic theory, and to present an ergodic theorem for multivalued operators.

D:14

A FIXED POINT THEOREM IN PROBABLISTIC METRIC SPACES

R. C. Dimri and U. C. Gairola

Department of Mathematics, HNB Garhwal University, Srinagar,
Pauri Campus, Pauri Garhwal

In paper we introduce the concept of R-weak commutativity for a pair of selfmap in probablistic metric space and establish a fixed point theorem. Our result is a generalization of many well known previous results.

D:15

MATRIX TRANSFORMATIONS ON SEQUENCE SPACE l_π

K. Chandrasekhara Rao and N. Subramanian

Department of Mathematics, Alagappa University, Karaikudi

Let (π_k) be a sequence of positive numbers. Let $l_\pi^{\propto v}$ be the space of all complex sequence (x_k) such that $\left(\frac{x_k}{\pi_k}\right)\in l$. For each fixed positive integer k, s^k denotes the sequence $(0, 0, \ldots, \pi_k, 0, \ldots)$, π_k in the k-th place and zero at other places. First we show that the set $E=\{s^k : k = 1,2,\ldots\}$ is a

determining set for l_π. The set of all infinite matrices transforming l_π into a *BK*- space *Y* is denoted by (l_π: *Y*). We characterizes the classes (l_π: *Y*) when $Y=l_\infty$, c_0, c, l^p ($p\geq1$), bv, $b\upsilon_0$, bs, cs, l_p. In summary we have the following table

↗	l_∞	c_0	c	l^p $1 \leq p < \infty$	bv	bv_0	bs	cs	l_p
l_π	Necessary and sufficient condition on the matrix are obtained.								

D:16

COINCIDENCE AND FIXED POINT THEOREMS FOR A FAMILY OF MAPPINGS ON MENGER SPACES

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Using an idea developed by M. Stojakovic, we generalize some results of Singh and Pant.

D:17

SOME RESULTS IN FIXED POINT THEORY AND BEST APPROXIMATIONS

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D:18

FIXED POINT ITERATION PROCEDURES

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D:19

PARTITIONS OF UNITY AND APPLICATIONS

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D:20

STABILITY OF ITERATIVE PROCEDURES FOR BANACH TYPE OPERATORS

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The intent of this note is to present stability theorems for Banach type operators in *b*-metric spaces. Our results extend a known similar results in metric spaces.

D:21

BEST SIMULTANEOUS APPROXIMATION AND SOME GENERALIZED NORMS OF *TYPE A* AND *TYPE M*

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In the present paper we have dealt with some of the problems in best simultaneous approximation under some generalized *type A* and *type M* norm conditions and their applications. Some generalized orthogonality conditions of the type introduced recently by Dragomir have also been used in the process. Our results in some way extend; some of the earlier results obtained in the same direction by Benitez and Fernandez.

D:22

AN OLD PROBLEM OF SAMUELSON AND THE JACOBIAN CONJECTURE

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Suppose F is a C^1 - differentiable map from $\Omega \subset R^n \rightarrow R^n$, what condition should we impose on Ω and F such that F is (globally) one-to-one on Ω. Let J stand for the Jacobian of the map F. Non-vanishing of J over Ω need not imply one-to-one of F in general. This problem arises naturally in Mathematical Economics in connection with "Factor price equalization" considered by the well-known economist and Nobel-Lawreate, Paul Samuelson. We will discuss this problem in some depth and relate it to another well-known conjecture called "The Jacobian Conjecture"

D:23

EXTENSION OF RHOADE'S STABILITY RESULT

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The intent of this paper is to extend a few stability result of B. E. Rhoades to b-metric spaces.

E:1

EXISTENCE OF SOLUTIONS OF SECOND ORDER SEMILINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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In this paper we prove the existence of solutions for semilinear second order neutral functional differential equations using the theory of strongly continuous cosine families. The result are obtained by using the Schaefer theorem.

E:2

EXISTENCE OF SOLUTIONS OF NEUTRAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATION IN BANACH SPACES

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In this paper we prove the existence of solutions for neutral functional integrodifferential equation in a Banach space. The result are obtained by using the Schaefer fixed point theorem.

E:3

ON SOME THEOREMS RELATED TO A SYSTEM OF SIMULTANEOUS DIFFERENTIAL EQUATIONS

S. N. Singh and U. S. Metha

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In this paper some theorems regarding Wronskian of the solutions associated with a system of simultaneous differential equations have been proved.

E:4

NONOSCILLATION OF NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATION

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Some sufficient conditions for the existence of nonoscillatory solutions of a nonlinear neutral differential equation are obtained by using fixed point technique. An example is provided to illustrate the theory.

E:5

A SECOND ORDER DIFFERENTIAL SYSTEM WITH NON SMOOTH COEFFICIENTS

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We consider the Dirichlet problem

$$\mathcal{L}\phi=0,\ \phi(0) = \phi(1) = 0$$

assuming that it possesses only the trivial solution,

where

$$\mathcal{L} \equiv \begin{pmatrix} -\frac{d}{dx}\left(p_1\frac{d}{dx}+p_2\right)+q_1\frac{d}{dx}+r_1 & q_2\frac{d}{dx}+r_2 \\ q_2\frac{d}{dx}+r_2 & -\frac{d}{dx}\,p_3\ \frac{d}{dx}+p_4\,)+\ q_3\frac{d}{dx}+r_3 \end{pmatrix}$$

$\phi = (u, v)^T$, p_1, $p_2 \in L^\infty(0,1)$, p_2, p_4, q_1, q_2, $q_3 \in L^2(0,1)$, r_1, r_2, $r_3 \in L^1(0,1)$, p_1, $p_3 \geq \alpha > 0$ almost everywhere on $(0,1)$; $\alpha \equiv$ constant and show that the boundary value problem

$$\mathcal{L}\,\phi = f = g,$$
$$\phi(0) = \phi(1) = 0$$

has a unique solution ϕ belonging to the sobolev space $H_0^1(0,1)$.

E:6

GLOBAL EXISTENCE OF SOLUTIONS OF NONLINEAR FUNCTIONAL INTEGRAL EQUATIONS

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In the present paper global existence resutls for certain nonlinear functional integral and integrodifferential equations are established. Our analysis is based on an application of the Leray-Schauder alternative.

E:7

EXISTENCE OF SOLUTIONS OF QUASILINEAR INTEGRO-DIFFERENTIAL EQUATION WITH DEVIATING ARGUMENTS

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In this paper we prove the existence of mild solution for quasilinear integrodifferential equation with nonlocal condition. The results are established by the method of compact semigroups and the Schaefer theorem.

E:8

EXISTENCE OF WEAK POSITIVE SOLUTION OF COUPLED HAMMERSTEIN INTEGRAL EQUATIONS VIA FIXED POINT TECHNIQUE

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In this paper a coupled fixed point theorem concerning the solution of the operator equations $\lambda x=T(x,y)$, $\lambda y=T(y,x)$ is proved in Banach spaces which is further applied to the coupled Hammerstein integral equations.

$$\lambda x=(t) = \int_J k(t,s)\, f(s,x(s),y(s))\, ds$$

and

$$\lambda t=(t) = \int k(t,s)\, f(s,y(s), x(s))\, ds$$

for proving the existence of the coupled positive solution under suitable conditions. To illustrate the abstract theory developed here, an application is given to the nonlinear coupled boundary value problems of ordinary differential equations for proving the existence of coupled positive solution.

F:1

SOME THEOREMS CONCERNING PSEUDO ANALYTIC VECTORS ON PSEUDO-KAEHLERIAN MAINIFOLDS

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In the present paper, we have defined pseudo-analyticity and have investigated some theorems which concern pseudo-analytic vectors on pseudo-Kaehlerian manifolds.

F:2

ON GENERALISED RECURRENT RIEMANNIAN MANIFOLDS

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In this paper we have studied on a generalized recurrent Riemannian manifold using two 1-forms A and B. It is shown that for a non-zero constant scalar curvature, A is closed if B is closed for a non-constant scalar curvature the 1-forms A and B can not be closed both, unless A is collinear with B. Also, we have found out some results on a generalised conformally recurrent Riemannian manifold.

F:3

ON AFFINE CONNEXIONS IN AN ALMOST HYPERBOLIC CONTACT MANIFOLD

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In this paper we define several affine connexions in an almost hyperbolic contact manifold and study some of its properties.

F:4

ON COSYMPLECTIC MANIFOLD WITH CONTACT BOCHNER CURVATURE TENSOR

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In this paper we have duscussed Cosymplectic manifold M_n (n=2m+1), possessing the orthonormal basis $\{e_1, e_2, e_3,$ -----------$, e_{2m}, Fe_1, Fe_2, Fe_3,$------$, Fe_{2m}\}$ of unit vectors which are normal to the contact vector. If we have obtained the expressions relating the sectional curvature and the scalar curvature in this manifold with vanishing contact Bochner curvature tensor.

F:5

ERROR CORRECTING CODES, MATROIDS AND ALGEBRAIC GEOMETRY

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We shall discuss a basic problem in the theory of error correcting codes, which admits equivalent formulations in terms of uniform matroids as well as Grassmannians over finite fields. After reviewing the earlier work on this problem, we shall describe some recent joint work with G. Lachaud. Applications to other aspects of Coding Theory as well as to Algebraic Geometry will also be discussed. An attempt will be made to keep the prerequisites at a minimum.

G:1

COINCIDENCES AND FIXED POINTS OF NONEXPANSIVE TYPE MAPS

S. L. Singh, D. D. Sharma and S. S. P. Singh

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Fixed point theory of nonexpansive and nonexpansive type maps provides techniques for solving a variety of applied problems in mathematical sciences and engineering. The purpose of this paper is to study nonexpansive type maps studied by M. Chandra, S. N. Mishra, S. L. Singh and B. E. Rhoades [Indian J. Pure Appl. Math 26 (1995), 393-401] on the setting of 2-metric spaces.

G:2

A UNIFIED COINCIDENCE THEOREM

S. L. Singh, D. D. Sharma and S. S. P. Singh

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In this paper we shall obtain a general coincidence theorem for Matkowski type operators on a finite product of 2-metric spaces. Several coincidences and fixed point theorems are obtained as special cases of our theorem.

G:3

INTUITIONISTIC GRADATION OF OPENNESS : INTUITIONISTIC FUZZY TOPOLOGY

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Let X be a nonempty set. We define an intuitionistic gradation of openness of fuzzy sets of X as an ordered pair (τ,τ^*) of functions form I^X to I such that

(1) $\tau(\lambda)+\tau^*(\lambda)\leq 1, \forall \lambda \in I^X,$

(2) $\tau(\tilde{0})+\tau(\tilde{1})=1;\ \tau^*(\tilde{0})=\tau^*(\tilde{1})=0,$

(3) $\tau(\lambda_1\cap\lambda_2) \geq \tau(\lambda_1)\wedge\tau(\lambda_2);\ \tau^*(\lambda_1\cap\lambda_2) \leq \tau^*(\lambda_1)\vee\tau^*(\lambda_2),$

(4) $\tau(\cup_i\lambda_i) \geq \wedge\ \tau_i(\lambda_i);\ \tau^*(\cup_i\lambda_i) \leq \vee_i\tau^*(\lambda_i).$

The triplet (X, τ, τ^*) is called an intuitionistic fuzzy topological space. With this definition of an intuitionistic fuzzy topological space, we prove that the category of intuitionistic fuzzy topological spaces and gradation preserving mappings from a topological category. We also study compactness of intuitionistic fuzzy topological spaces and prove Tychonoff theorem.

G:4

ON λ-CONTINUOUS MAPS IN TOPOLOGICAL SPACES

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Several researchers in topology have introduced several weaker forms of closed sets and continuous maps in topological spaces. In this paper, we introduce a new concept called "λ-closed set" which is defined as follows : *A* subset *A* of *a* topological space *X* is called λ-closed if *A* contains the closure of *G* whenever *A* contains *G* and *G* is open. We observe that λ-closed set is weaker than closed set. Using λ-closed set, we introduce λ-continuous maps. Some properties of the newly introduced concepts have been investigated.

G:5

ON GENERAL, HYBRID CONTRACTIONS

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Khan *et al*. have studied the existence of solutions of functional equations $f_i x \in Sx \cap Tx$ and $x = f_i x \in Sx \cap Tx$ under certain nonlinear hybrid contractions and asymptotic regularity conditions, generalizing many known results on coincidences and fixed points. However, most of their main theorems admit counter examples. In this paper, we have rectified these results and have obtained many coincidence and fixed point theorems rather on a more general setting.

G:6

FIXED POINT THEOREMS FOR JUNGCK TYPE HYBRID CONTRACTIONS

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The purpose of this paper is to show the existence of a common fixed point for triplets of maps (P,Q,f) and (T,F,g) satisfying

$$\delta(Px, Qy) \leq q.\max\{d(fx, fy), H(fx, Px), H(fy, Qy), \tfrac{1}{2}[H(fx, Qy)+H(fy, Px)]\}$$

and $$\delta(Tx, Ty) \leq q.\max\{d(fx, gy), H(fx, Tx), H(gy, Ty), \tfrac{1}{2}[H(Fx, Ty)+H(gy, Tx)]\}$$

for all x, y in X, wherein (X,d) is a metric space, H is the Hausedorff metric induced by d, $\delta(A, B) = \sup\{d(x, y) : x \in A, y \in B\}$, $P, Q, T : X \to CB(X)$, $f, g : X \to X$, $0<q<1$ and $CB(X)$ is the collection of all nonempty closed and bounded subsets of X.

G:7

COINCIDENCE THEOREMS FOR FOUR MAPS

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Department of Mathematics, CDC Hardwar 249 404

The purpose of this paper is to obtain coincidence theorems for four maps on an arbitrary set with values in a metric space. As a consequence, results of Singh *et al.* are obtained as special cases.

G:8

MULTI-VALUED MAPPINGS WITH COMMON INVARIANT POINTS IN 2-METRIC SPACES

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The notion of 2-metric space was first introduced by S. Gähler in 1963/64 and since then it was extensively explored by among others, C. Diminnie, A. G. White, K. Iseki and B. E. Rhoades. In the sequal following Banach contraction principle Jr. S.B. Nadler introduced the concept of multi-valued contraction mapping and established that it possesses a fixed point in a complete 1-metric space. Later on Chang and Huang in 1989 proved some fixed point theorems for set-valued mappings in 2-metric spaces. This paper intends to generalize the corresponding results.

G:9

FIXED POINT OF EXPANSION MAPPINGS ON RANDOM 2-NORMAL SPACES

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In this paper, the concept of expansion mappings on a random 2-normal space has been introduced and some fixed point theorems for such mappings have been established. There results being new of their kind, are interesting enterious of some known results.

G:10

D-CONTINIOUS FUNCTIONS

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In the present paper we investigated some theorem which concern D_i- continuity on general topological space.

G:11

NONLINEAR CONTRACTION IN *D*-METRIC SPACES

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In this paper a fixed point theorem is proved for the self-maps f of a D-metric space X satisfying the nonlinear contraction condition, namely.

$$\rho^r(fx, fy, fz) \leq \phi\,(\rho^r(x, y, z) \qquad (*)$$

for all $x, y, z, \in X$, $r \in N$, where $\phi : R^+ \rightarrow R^+$ is a continuous nondecreasing function satisfying $\phi(t) < t$, $t > 0$.

Our result generalizes the fixed point theorem of Dhage [Bull. Cal. Math. Soc. 84(4) (1992), 329-336] and Dhage and Rhoades [Epsilon Jorn. Math (accepted)].

G:12

TOPOLOGICAL QUANTUM FIELD THEORY

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H:1

MINIMUM ENTROPY PROBABILITY DISTRIBUTION

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An analytical expression for the minimum entropy *Smin* as a function of prescribed value *m* of any mean (arithmetic, geometric, harmonic, power) is obtained when the random variate takes values x_1, x_2,........,x_n; where $0 < x_1 < x_2 <$........x_n It is shown that (1) in the i^{th} subinterval (x_i, x_{i+1}) i=1,2.........,n-1; *Smin-m* curve consists of two arcs. One starting from x_i with zero value and increasing to a maximum value (*Smin*) *max* at yi $(x_i < y_i < x_{i+1})$ and other starting from the value (*Smin*) *max* at y_i and decreasing to a value zero at x_{i+1}.

(2) *Smin-m* curve is continuous throughout, but is not differentiable at $y_1, y_2, \ldots\ldots, y_{n-1}$.

(3) $$\left[\frac{Smin\text{-}m}{dm}\right]_{x_i+0} = \infty, \qquad \left[\frac{d\,Smin}{dm}\right]_{y_i-0} > 0\,,$$

$$\left[\frac{d\,Smin}{dm}\right]_{y_i-0} < 0\,, \qquad \left[\frac{d\,Smin}{dm}\right]_{x_{i+1}-0} > -\infty\,.$$

and $\left[\frac{d\,Smin}{dm}\right]$ does not vanish for any value of x.

(4) *Smin-m* curve has cusps at x_2, x_3,........,x_{n-1};.

(5) The n-1 values of (*Smin*) *max* in the intervals are equal if arithmetic (geometric, harmonic, r^{th} order power) mean is prescribed and x_1, x_2,........,x_n; (or inx_1,inx_2,.........inx_n; $1/x_1$,$1/x_2$,........, $1/x_n$; x_1^r,........, x_n^r) are in arithmetic progression.

I:1

ON THE RATE OF CONVERGENCE BY MODIFIED SZASZ – MIRAKYAN OPERATORS

Vijay Gupta and Prabhakar Gupta

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Recently Gupta and Srivastava (1995) introduced a new kind of Szasz-Mirakyan operators by taking the weight functions of Baskakov operators, so as to approximate Lebesgue integrable functions on $[0,\infty)$ interval. In the present paper, we extend the result of Gupta and Srivastava, we obtain a Voronovskaja type asymptotic formula and an estimate of error for the linear combinations of these summation-integral type operators.

I:2

RATE OF CONVERGENCE OF MODIFIED BASKAKOV TYPE OPERATORS OF FUNCTIONS OF BOUNDED VARIATION

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Recently Aniol and P. Pych Taberska [Approximation Theory and its Appl. 11:2 (1995), 94-105] obtained the rate of pointwise convergence of some Durrmeyer type summation - integral operators. Gupta and Kumar [Demonstratio Mathematica *XXX* No.2 (1997) 339-346] used some results of probability theory and they estimated the rate of convergence for Baskakov-Durrmeyer operators of functions of bounded variation. In the present paperwe obtain the rate of pointwise convergence for recently introduced integrated Baskakov type operators, We have improved the earlier results of Guo and Khan [J. Approx. Theory 58 (1989), 90-101] and Aniol and Taberska (1995).

I:3

ON THE INTERPOLATION SERIES OF ENTIRE FUNCTIONS

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The paper deals with coefficients characterization of lower (p, q)-order and generalized lower (p, q) - type of Newton's interpolating series of entire functions. Finally, some inequalities connecting generalized (p, q)- type and generalized lower (p, q)-type fo interpolating series have been drived.

I:4

RELIABILITY TREES-A NEW CONCEPT FOR PERFORMANCE EVALUATION OF A COMPUTER COMUNICATION NETWORK

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Overall reliability of a Computer Communication Network (*CCN*) is one of the most important criteria for the determination of its performance. In case of a *CCN* having all duplex links, the concept of the spanning trees formed the basis of most of the researches carried out for the evaluation of network overall reliability. the conventional definition of spanning trees for *CCNs* involving some (or all) simplex links given as "A set of links which is free from cycles and connects all the nodes of the network", does not provide both way communication paths between each pair of nodes of the network. Therefore, no research has so far been carried out for evaluating overall reliability of a *CCN* involving simplex links. The aim of this paper is to introduce a new concept of reliability tree (RT) for evaluating overall reliability of a *CCN* having some oriented channels. Enough examples have been included in the body of the paper to explain the concept.

I:5

ON THE DEGREE OF APPROXIMATION BY SPECIAL POSITIVE LINEAR OPERATORS

A. K. Sao, R. K. Gouraha, A. S. Ranadive, P. Verma and S. P. Singh
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In the present paper, some approximation properties on a special positive linear operators defined as :

$$(L_n f)(x)=\frac{1}{\varnothing_n(x)}\sum_{k=0}^{\infty} a_n, k\, x^k\ \frac{n^{\lambda+k+1}}{\left|\underline{\lambda+k+1}\right.}\int_0^{\infty} e^{-nt}\, t^{\lambda+k}\, f(t)\, dt$$

have been discussed. (Here $o_n(x)$; a_n, k are some suitable choices of functions defined in the paper).

I:6

LEE DISTANCE AND NEGACYCLIC CODES

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The properties of the BCH (Base-Chavdhuri- HOCQuenghem) codes have been extensively investigated. Negacyclic code is a form of BCH code and utilises the concept of Lee distance whose properties have been stated.

I:7

DECIDING LINK PRIORITIES IN THE DESIGN OF A COMPUTER COMMUNICATION NETWORK FOR TERMINAL RELIABILITY OPTIMIZATION

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Building reliability into the design of a Computer Communication Network (*CCN*) is the need of the hour which depends on the reliabilities of the network elements and its topological design. The topological design is basically concerned with location of computers and communication links connecting them so as to satisfy the performance requirements with the least cost. The problem of preparing a topological layout of a *CCN* with optimum *ST*-reliability is faced by the network designer whenever a new network is set up or an old network is expanded. The present paper attempts to address the above problem by suggesting a heuristic approach for the same. The goal is achieved by deciding the physical location of computers and communication links in such a way that the network required for desired connectivity is optimized. Thus the *ST*-reliability of a *CCN* with a fixed topology is optimized for a given set of link-reliability of a *CCN* with a fixed topology is optimized for a given set of link-reliabilities. The results reported in this paper have been compared with those derived from exhaustive enumeration technique. It is observed that these results are very close to the true optimum results in all the cases.

I:8

ON L_1-APPROXIMATION BY THE GENERALISED SZÃSZ - MIRAKJAN OPERATOR

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The object of the paper to give the degree of L_1 approximation to an integrable function f by the generalised Szãsz Mirakjan operator defined as

$$(L^{*}_{n+p} f) = (n+p)\sum_{k=0}^{\infty} e^{-(n+p)} \times \frac{(n+p)^k x^k}{1.2.....k}\int_{k/n}^{(k+1)/n} f(t)\,dt$$

for $n+p \geq 0, p \geq 0, f \in L_1[0,\infty)$

involving the integral modulus of continuity $\omega(f;\delta)$ in L_1 norms with the density function $\sqrt{x}$.

I:9

RESPONSE TIME ANALYSIS OF PAKET SWITCHED NETWORK

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For transmission of data beyond a local area, communication is tyorically achieved by transmitting data from source to destination through a network of intermediate switching nodes. The switching nodes are not concerned with the content of the data, rather their purpose is to provide a switching facility that will move the data from node to node as it reaches its destination.

In the present paper the development of an appropriate queuing model is presented for the performance evaluation of the packet switching in data communication. It consists of a transmitter and a receiver connected together with a communication channel in a packet switched network. A pair switching of nodes can be considered as combination of a transmitter and a receiver. Arrival of the message at the transmitter is assumed to follow Poisson distribution. Main aspect of the model is priority service schemes of transmitters. Two types of messages are assumed with arrival rates λ_1 and λ_2 respectively coming to a transmitter. If λ_1 message has priority over λ_2 then transmitter transmits first λ_1 then λ_2 .The result for mean waiting time at the transmitter, mean transmission time & mean service time of receiver have been derived to determine the mean response time of the complete system. The result are depicted in both tabular as well as in graphical forms.

I:10

STRONG UNIQUENESS OF BEST SIMULTANEOUS APPROXIMATION

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For investigating strong uniqueness of best simultaneous approximants, one considers here three properties of the triples $(X,V,\mathcal{F})$ where X is a normed linear space, V is a nonempty closed subset of X and $\mathcal{F}$ is a family of nonempty closed and bounded subsets of X. These properties ae called *SUBSA*, α-*SUBSA* and *SUBSA* of order q. Examples satisfying these properties are explored. In particular, the classical Haar theory for $C_0(T)$ as well as the more recent Haar theory for the function space $C_0(T,H)$ of Hilbert space-valued continuous functions are extended to best simultaneous approximants of certain sets.

I:11

OPTIMIZATION MODEL FOR RELIABILITY INDEX BASED TASK ALLOCATION IN DISTRIBUTED SYSTEMS

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The reliable performance for a Computer Communication Network (CCN) under various conditions is of utmost importance in many industrial, military and everyday life situations. Many desirable services (e.g. file service, authentication service in distributed system need be dealt with highly fault - tolerant and secured manner. Therefore, it is desirable to increase the reliability and security of services by destributing the responsibility for providing the service among many servers. There are many ways to increase the performance of a distributed system. A common approach is to distribute the processes and computer resources in such a way that the overall execution time is optimized. The present paper aims at developing a reliability - based optimal task allocation model for distributed systems. First we suggest a reliability and cost based index, based upon which overall task allocation reliability is optimized. The developed algorithm is coded into *FORTAN*-90 and implemented on *RISC* based *DEC*-2000 machine. The time complexity of the algorithm is observed to be $O(n^m)$.

I:12

SOME APPROXIMATION RESULT ON A POSITIVE LINEAR OPERATORS

P. Verma and A. K. Sao, R. K. Gouraha, A. S. Ranadive and S. P. Singh
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In the present paper we define a sequence of positive linear operators as :

$$(L_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \mathrm{p}_k(nx) \frac{n}{dk(1)} \int_0^{\infty} p_k(nt)\, e^{-nt} (S_n f)(t)\, dt \qquad 1.1$$

where $(S_n f)(t) = e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{\underline{|k}} f(\frac{k}{n})$ and $p_k(nx)$ and $dk(x)$

are some suitable functions defined in the paper. We prove some approximation results on the operators and deduce some result on the corresponding operators also.

I:13

SOME APPROXIMATION PROPERTIES ON A BASAKOVA TYPE OPERATOR

A. K. Sao, R. K. Gouraha, A. S. Ranadive, P. Verma and S. P. Singh
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In the present paper we define a sequence of a Basakova type operators, approximating a function $f \in L_1[c, \infty)$ as,

$$(L_n f)(x) = \sum_{k=0}^{\infty} \frac{(-x)^k \phi_n(x)}{\underline{|k}} \frac{n}{dk(t)} \int_0^{\infty} e^{-nt} p_k(nt) f(t)\, dt$$

where $\phi_n(x)$ and $dk(t)$ are some suitable choice of functions defind in the paper. We prove some approximation results on the operators and deduce some results on the corresponding special operators also.

I:14

ON THE DEGREE OF L_1-APPROXIMATION BY THE METHOD OF INTEGRAL BASKAKOV OPERATORS

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The object of the paper to give the degree of L_1 approximation to an integrable function f by the method of integrable Baskakaov operators

$$(L_n f)(x) = n\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{x^k}{(k+x)^{n+k+1}}\int_{k/n}^{k+1/n} f(t)\,dt$$

involving the integral modulus of continuity $\omega(f;\delta)$ in L_1 norms with the density function $\sqrt{x(1+x)}$.

J:1

NONDOMINATED SOLUTIONS OF A SITE LOCATION PROBLEM WITH TWO OBJECTIVE WITHOUT ASSIGNMENT OF PRIORITIES TO THEM

Satya Prakash

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The problem of selecting upto a fixed number of sites from among a given number of potential facility sites and asigning customers to sites in such a way that each customer is assigned to a unique selected site, is considered. The problem has two objective without assignment of priorities to them. One objective is to minimize the sum of the distances and the other objective is to minimize the maximum distance among the distances from each customer to his/her assigned site. A heuristic method is developed to find the set of nondominated solutions of the bicriterion site location problem.

J:2

A FUZZY CONTROL SYSTEM FOR TRAFFIC LIGHTS MANAGEMENT

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Fuzzy logic has found widespread applications in many diverse fields including artificial intelligence, engineering, operations research, robotics and pattern recognition. In situations, where information required for taking decisions is not complete or its reliability questionable. Fuzzy logic has proved immensely beneficial. In this paper a Fuzzy Control System has been applied to manage Traffic Lights by demarcating the traffic density into various fuzzy categories such as Light, Medium and Heavy for an inference engine to analyse the output provided by the Control System to take necessary appropriate actions. This program can

be used in situations that involve similar kinds of fuzziness in general and to traffic management in particular. The application to this program will ensure smooth flow of traffic thereby helping in the day to day management of vehicular traffic.

J:3

AN INVENTORY MODEL FOR BANKING SECTOR TO CONTROL CREDIT

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When Central Bank reduces the credit rate of interest then Commercial Bank might place a large order and offer a reduced credit rate of interest to customers. An inventory model is developed to control credit in banking industry. The optimal order quantity and profit are determined for commercial Banks. The monetory phenomenon in banking sector is explained by sensitivity analysis results.

J:4

A SINGLE PERIOD STOCHASTIC INVENTORY MODEL WITH ERLANG DEMAND DISTRIBUTION

R. M. Bhandari
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A single period inventory problem with Erlang demand distribution is considered. The model contains the usual ingredients of a newsboy problem. Sale price, purchase cost, salvage value and shortage cost. In addition, there is a cost per unit of marketing effort, and the demand distribution is stochastically increasing in that effort. It is assumed that the distribution of demand can be shifted up by an increase in sales effort. The paper deals with the simultaneous determination of optimal order quantity and sales effort. The optimum profit associated with optimum marketing effort is also determined.

J:5

ANALYSIS OF A PARALLEL TRANSIT SYSTEM UNDER HEAD OF LINE REPAIR PRIORITY

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This paper considers a system consisting of two subsystems connected in series with a single repair facility. One subsystem is *j* out of M : G system consisting of M identical units, while the other consists of N non-identical units arranged in series. The life times of the active units depends on each other in having simultaneous failure of all the operating units and repair times are distributed quite generally. The system breaks down completely if more than M-j+1 units in the parallel group are simultaneously in the failed states or if any failure occurs in the series group of if any failure occures due to critical human error. The availability and reliability function of the system are obtained simultaneously. Explicit expressions for the steady state availability of the system and the mean time to the first system failure are obtained.

J:6

SYMETRIC DUALITY IN VARIATIONAL PROBLEMS WITH F-CONVEXITY

S. K. Mishra

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Under F-convexity, F-concavity F-pseudoconvexity, F-pseudoconcavity, appropriate duality results are established for Wolfe and Mond-Weir type symmetric dual variational problems. Using these results duality theorems for fractional variational problems are also obtained under additional condition on the kernel function above pair of problems are exibited to be selfdual.

J:7

QUEUEING NETWORK MODEL FOR MULTIPROGRAMING

Arif Nadeem and Maneesh Kumar Singh

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In this paper we present two different models for multiprogrammed computer system. The first takes advantage of the original Gordon and Newell closed queueing network model and therefore requires the use of exponential service times throught. The second model makes use of diffusion approximation which permits the use of arbitrary distributions but yields only approximate answers.

J:8

RELIABILITY PARAMETERS OF A COMPLEX SYSTEM UNDER WAITING AND MIXED FAILURE PATTERN

P. P. Gupta and Ekata

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The mathematical model under consideration consisting of n-units in series suffers multitypes of failures. The system has three modes; viz. normal, degraded & failed with hardware and human failure. All types of failure and waiting rates are exponential while repair rates are distributed quite generally. System state probabilities and other parameters are developed for the proposed model using supplementary variable technique.

J:9

DISTRIBUTED SERVICE NETWORKS : SEPARATE DISPATCHING POLICIES

Arif Nadeem

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This paper exhibits two distinct service network, and develops a dispatching policy for each under no cooperation conditions. These policies will serve as a basis for comparision with cooperation policies presented in the paper. The networks that have been chosen are very simple in order to clarify the exposition; however, an expansion to a more realistic case is quite straight forward.

J:10

MODEL FOR SINGLE LEVEL LOTSIZING AND SCHEDULING

Arif Nadeem and Amit Kumar Awasthi

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In the short term planning the sequence of the production quantities is required. This paper deals with models where the size and the sequence of lots are datermined simultaneously. These include the models for single-level, multi-item, dynamic, capacitated lotsizing and scheduling of the discrete and continuous set up respectively. Although these types of models are based on different fundamental assumption, each of them is applicable for short-term scheduling with periods as days or shifts.

J:11

A MATHEMATICAL MODEL FOR JOB SHOP SCHEDULING PROBLEMS

Arif Nadeem and Saurabh Srivastava

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This paper investigates the application of the algorithms available for solving the job shop scheduling problem. We determine the most powerful solution procedure for n jobs m machines problem. Our results show that the algorithms presented in this paper are either comparable to or better than the best algorithms for job shop scheduling problems. Most importantly, they represent a novel approach to solution improvement which holds promise in many job shop scheduling problems.

K:1

DIFFUSION OF CHEMICALLY REACTIVE SPECIES IN VISCO-ELASTIC FLUID FROM A STRETCHING SHEET

Ravi. M. Sonath and M. Subhas Abel

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This paper presents the solution of diffusion of chemically reactive species in the laminar boundary layer flow over an elastic plane surface in a visco-elastic fluid. The visco-elastic flow is driven soley by the linearly stretched surface and the reactive species is emitted from this sheet and undergoes an isothermal and homogeneous one-stage reaction as it diffuses into the surrounding fluid. Asimlarty transformation is devised, which reduces the non-linear concentration conservation equation to ordinary differtntial equation.

K:2

APPLICATIONS OF FUZZY LOGIC TO CLOSED LOOP

Sanjay Singh

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This paper opens with some fundamentals of fuzzy set and its extension to fuzzy logic. Further, how one can be fit to be fuzzy and fuzzy model have been discussed. This is followed by a description and an example of a fuzzy controller for a steam turbine and a design methodology for a fuzzy system.

K:3

HEAT TRANSFER IN *MHD* VISCO-ELASTIC FLUID FLOW OVER A STRETCHING SURFACE

M. Subhash Abel, K. V. Prasad and Ambuja Joshi

Department of Mathematics, Gulbarga University, Gulbarga 585 106

In this paper we considered the study of heat transfer characteristics in *MHD* visco-elastic fluid over a stretching sheet with variable wall temperature and internal heat generation.

Two cases are considered (i) The sheet with prescribed surface temperature (*PST*) (ii) The sheet with prescribed wall heat flux (*PHF*)

The effect of visco-elasticity on heat transfer is to accelerate wall temperature gradient in both *PST* and *PHF* cases. The wall temperature gradient will have positive values for all values of prandtl number which is a reverse effect compared to *MHD* viscous fluid flow past a stretching sheet as discussed by Vajravelu and Rollins [8].

K:4

CRITICAL LEVELS FOR GRAVITY WAVES

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Earlier studies on the critical levels for gravity wave propagation have established the possible existence of two type of critical levels.

i. The wave trappers which absorb waves that approach them irrespective of the direction in which the wave is approaching the critical level.

ii. The other type of critical levels, valve like critical levels, permit free transmission of waves across these levels without any attenuation while the same wave if approaches the critical level from the otherside of it is absorbed at the critical level.

The present study has established that first type of critical levels, the wave trappers are again valve like critical levels superimposed by a reflexion level. This in essence establishes that there exists only one type of critical level, that is, the valve like critical level for gravity wave propagation.

K:5

ANHOLONOMIC DESCRIPTION OF TWO + PHASE *MFD* FLOWS IN A ROTATING COORDINATE SYSTEM

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The equations governing the two-phase *MFD* flows has been written for steady incompressible flows. With one ignorable coordinate in orthogonal curvilinear coordinate system the solutions have been found out. The vortex-line triad has been employed to discuss the complex lamellar and Beltrami motions. It has been shown that for Beltrami two-phase magnetofluids with flat field restrictions, the Bernoulli surfaces contain the vortex lines if and only if the component of the viscous force along the vortex lines is equal to the component of force exerted by the particles on the fluid along the vortex lines.

K:6

TOWARDS QUASI ISOTROPIC STRESS-HEAT FLUX TURBULENCE MODEL FOR BUOYANCY DRIVEN FLOWS

A. K. Upadhyay and T. Dixit

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Department of Mathematics, K.N. Govt. P.G. College, Gyanpur 227 304

The theoretical relation between the temperature-pressure gradient correlation and the pressure-strain correlation is discussed in context with the quasi isotropic turbulence model. It is shown that for such a field the modelling of the heat flux transport equation becomes consistent with the modelling of the Reynolds stresses transport eqations.

K:7

EFFECTS OF THE DEPENDENT SURFACE INFILTRATION ON WATER TABLE VARIATION IN A SEMICONFINED AQUIFER : AN ALTERNATIVE ANALYTICAL SOLUTION

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An analytical solution for transient water table variation in a semiconfined and finite aquifer with recharge from canals and time dependent surface infiltration is presented. The solution is obtained by Laplace transform technique as well as appropriate transformation. It is found that appropriate transformation approach is simpler then Laplace transform technique. Numerical values of water table variation are obtained by these methods, which are in close agreement.

K:8

NATURAL CONVECTION BETWEEN COAXIAL VERTICAL CYLINDERS PARTIALLY FILLED WITH A POROUS MATERIAL

Trishna Paul and A. K. Singh

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This paper presents laminar fully developed free convection flow between two coaxial vertical cylinders partially filled with a porous matrix when the cylinders are kept at different temperatures. The Brinkman-extended Darcy model is used is used to simulate momentum transfer in the porous region. The two regions are coupled by equating the velocity and by considering shear strees jump condition at the interface. The effect of Darcy number on the velocity throughout porous domain and adjustable coefficient in the stress jump condition at the interface is shown graphically. It is observed that velocity is influenced by the shear stress jump condition at the interface.

K:9

NATURAL CONVECTION DUE TO HEAT AND MASS TRANSFER IN A COMPOSITE SYSTEM

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Department of Civil and Building Engineering, Victoria University of Technology,
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A numerical study is performed to analyse heat and mass transfer phenomena due to natural convection in a composite cavity containing a fluid layer overlying a porous layer saturated with the same fluid. The flow in the porous region is modelled using Brinkman-Forchheimar-extended Daray Model that includes both the effect of macroscopic shear (Brinkman effect) and flow inertia (Forchheimer effect). The vertical walls of the two dimensional enclosure are isothermal whilst the horizontal walls are adiabatic. The two regions are coupled by equating the velocity and stress components at the interface. The resulting coupled equations in non-dimensional form are solved by an alternating direction implicit method by transforming them into parabolic form by the addition of false transient terms. The numerical results show that the amount of fluid penetration into the porous layer depends strongly upon the Darcy, thermal and solutal Rayleigh numbers. Average Nusselt number decreases while average Sherwood number increases with an increase of the Lewis number. The transfer of heat and mass on the heated wall near the interface depends strongly on the Darcy number.

K:10

RAYLEIGH-BENARD CONVECTION IN A SECOND ORDER FERROMAGNETIC FLUID WITH SECOND SOUND

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The effects resulting from the substitution of the classical Fourier law by the non-classical Maxwell-Cattaneo law [1] in Rayleigh-Banard convection of a second-order ferromagnetic fluid are examined. The classical approach predicts an infinite speed for the propogatin of heat. The non-classical theory involves a wave type heat transport (SECOND SOUND) and does not suffer from the physically unacceptable drawback of infinite heat propogatin speed. The principle of exchange of stability is not guaranteed in the problem unlike the case of the Newtonian analog with Second Sound. It is found that the results are noteworthy at short times and the critical eigenvalues are less than the classical ones.

K:11

THE STABILITY OF DOUBLE DIFFUSIVE CONVECTION IN AN INCLINED FLUID LAYER WITH AND WITHOUT INTERNAL HEAT AND MASS GENERATION

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The linear stability theory is applied to examine the double diffusive convection that occurs in an inclined fluid layer confined between two parallel plates that are maintained at different temperatures and different concentrations. Using the normal mode approach that is slightly different, the amplitude functions are taken as functions of horizontal coordinate, we get a eighth order eigenvalue problem in which the differential equations involved are with variable cooffecients. The power series method is used to obtain the eigenvalue relation which can be solved numerically to obtain the exact stability conditions.

In the second problem, we obtain a similar eigenvalue problem of eighth order for double diffusive convection that occurs in an inclined fluid layer confined between parallel plates with uniformly distributed internal heat and concentration sources. It is assumed that the two plates bounding the layer are maintained at constant and equal temperatures. The power series metod is applied to obtain the equation for the eigenvalues. The stability conditions are obtained for various Prandtl numbers and Schmidt's numbers and various inclination angles. The stability sets in either as transverse travelling wave or as longitudinal stationary modes.

We solve the problems, in the particular cases, of horizontal and vertical layers and for longitudinal rolls.

K:12

THE EFFECT OF CORIOLIS FORCE AND NON-UNIFORM TEMPERATURE GRADIENT ON LINEAR AND NONLINEAR CONVECTION IN ISOTROPIC AND ANISOTROPIC POROUS CHANNELS - AN ANALYTICAL STUDY

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The effects of coriolis force and non uniform temperature gradient on the onset of convection in a horizontal rectangular channel filled with an incompressible fluid saturated by anisotropic porous media is studied by means of linear stability analysis. We derive the critical Rayleigh number expression analytically, when the walls of the channels are assumed to be impermeable and heat conducting to establish a linear temperature distribution in the vertical direction. The solution of the problem is saught in the form of Fourier series with its coefficients as functions of horizontal co-ordinate. We found that the effect of rotation in anisotropic porous permeability and thermal diffusivity have a significant effect on the critical Rayleigh number and the study flow patterns and the effect of rotation more often destablises the system. We have also carried out the non-linear stability analysis based on the Stuart Watson weakly non linear approach and determine the essential non-linear correction to the critical Rayleigh number and the corresponding solution.

K:13

MODELLING OF ACOUSTIC WAVE PROPAGATION INSIDE IRREGULAR WAVEGULIDES

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K:14

BINGHAM FLUID FLOW IN AN INCLINED CHANNEL BOUNDED BY TWO PERMEABLE BEDS

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Department of Mathematics, Sri Venkateswara University, Tirupati 517 502
Department of Mathematics, V.R. College, Nellore

The flow of a Bingham fluid in an inclined channel bounded by two permeable beds is investigated. The flow in the channel is according to Bingham model and the flow in the permeable beds is described by Darcy's law. The expressions for the velocity field, the pressure and the temperature distribution, the fractional increase in mass flow rate and the shear stress are obtained. The results are deduced and discussed.

K:15

VIBRATION OF NON HOMOGENEOUS INFINITE PLATE OF PARABOLICALLY VARYING THICKNESS CONSIDERING DAMPING EFFECT

Rajendra Kumar, D. C. Gupta and S. P. Kohli

Department of Mathematics, J.V. Jain (P.G.) College Sharanpur
Department of Mathematics and Statistics, GKV, Hardwar 249 404

In the present paper our investigations are devoted to the non homogeneous elastic bodies of prabolically varying thickness. The method used is frobenius for solving the problem.

K:16

TOWARDS QUASI ISOTROPIC STRESS-HEAT FLUX TURBULENCE MODEL FOR BUOYANCY DRIVEN FLOWS

T. Dixit, Anil Upadhyay and Dinesh Pratap Dubey

Department of Mathematics, K. N. Govt. P.G. College, Gyanpur 227304

The theoretical relation between the temperature-pressure gradient

correlation and the pressure-strain correlation is discussed in context with the quasi isotropic turbulence model. It is shown that for such a field the modelling of the heat flux transport equation becomes consistent with the modelling of the Reynolds stresses transport equations.

L:1

AXIALLY SYMMETRIC STATIONARY VACCUM FIELDS IN GENERAL RELATIVITY

M. D. Patel and R. M. Patel

Department of Mathematics, S.P. University, V.V.Nagar 388 120

A class of oxially symmetric stationary vacuum solutions which one of Petrov type N one obtained.

L:2

AXISYMMETRIC HYDROMAGNETIC STAGNATION FLOW OF A NEWTONIAN CONDUCTING FLUID

D. K. Phukan

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Axisymmetric hydromagnetic stagnation flow of a viscous, incompressible, electrically conducting fluid in the presence of a transverse magnetic fields is studied. A series solution of the similarity equation is presented for the case of large magnetic parameter. The Galerkin's approximate technique is used to solve the similarity equation expressed in terms of Crocco's variables. Then, an accurate expression the Shear Stress parameter is derived using Crocco's trnasformation. This is followed by a numerical solution of the similarity equation using a direct Runge-Kutta Shooting algorighm with Newton iteraction in double precision. Further more, the effects of the magnetic field on the velocity and Shear stress parameter, are analysed.

L:3

REMARKS ON A PAPER OF BANERJEE AND KATYAL

Harikishan and J. S. Parihar

Department of Mathematics, D. N. College, Meerut

Banerjee and Katyal (1988) established a result that, at the marginal state in the magnetohydrodynamic simple Benard problem, the total kinetic energy associated with a disturbance is greater than or equal to

its total magnetic energy in the regime $Q\sigma_1 \leq 3$ which is not correct. Here we have obtained the correct form of this result.

L:4

ELECTRONS IN CONDUCTORS

V. N. Sharma and S. R. Chaudhry

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This paper deals with the behaviour of electrons in conductors. Expression for the drift velocity is discussed. Individual and collactive roles are explained. For clear insight need of laser study is necessary.

L:5

POES TREATMENT OF DEPRECISSION

V. N. Sharma

Department of Mathematics, Govt. Degree College, Rudhauli, Basti

This paper deals with the poes treatment of deprecission.

L:6

MHD UNSTEADY FLOW OF A DUSTY VISCO-ELASTIC FLUID INDUCED BY THE MOTION OF A SEMI-INFINITE PLATE MOVING WITH VELOCITY DECREASING EXPONTIALLY WITH TIME

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This paper concerns with the *MHD* flow of a dusty viscoelastic fluid induced by prescribed motion of a semi-infinite plate. Expressions are obtained for the fluid phase and the dust-phase velocity distributions and also for the magnetic field. Some interesting conclusions are made.

L:7

COVARIENT FORMULATION OF SUPERFIELD LAGRANGIAN DENSITY IN "SUPER SPACE"

K. N. Baramola and D.C. Joshi

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Extension of 3-dimensional Eucleadian space to 4-dimensional Minkowski space led to unify the electricity and magnetism into a single electromagnetic field strength tensor. Various attempts have been made to extend 4-dimensional space further to unify all the four fundamental interactions i.e. electromagnetic, gravitational, strong and weak interaction. Super-space formulation is a similar attempt in 4+4=8 dimensional space.

In this paper the structure of a super-field is discussed in super-space. In addition to this, super-symmetry and super-symmetric transformations of component fields are given and finally the Lagrangian formulation is carried out for scalar and vector super-fields.

M:1

DYNAMICS OF HIV INFECTION OF *CD*4+*T*CELLS WITH TIME DELAY

Jonaki B. Ghosh

Department of Mathematics, Delhi University, New Delhi

We explore a model proposed by Alan S Perelson which describes the interaction of HIV with CD4+ T cells. The model comprises of four differential equations. Specifically four populations namely the uninfected T-cells, latently infected T-cells, actively infected T-cells and the free virus are considered in this model. It exhibits two kinds of steady states-the uninfected steady state where there is no virus present and the endemically infected steady state in which both virus and T- cells are present.

In this paper we propose a modified version of the model by introducing time lag. The model is based on the assumption that the interaction of the free virus and the T-cells has a delayed effect on all four populations. The terms in the differential equations of Perelson's model, indicating the interaction between the virus and T-cells are replaced by delay interaction terms with suitably chosen delay kernel function. Thus the differential equations of the model are replaced by intergrodifferential equations. we study the stability of both steady states by considering a distribured delay model and serveral combinations of delay kernel functions. An interestiong conclusion of the analysis is that an originally stable system remains stable after introducing time lag (irrespective of the choice of the delay kernel functions) whereas an originally unstable system may remain unstable or become stable depending upon the choice of the delay kernal function.

M:2

MATHEMATICAL MODEL FOR MASS TRANSFER IN AN ARTIFICIAL KIDNEY

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The object of the present investigation is to present analytical study of the mass transfer processes in a tabular hemodialyser. The dialysing fluid flows inside the tube and the blood flows over the tube in a thin film. The walls are supposed to be permeable. The attempt is made to estimate the redistribution of materials across the circular inter face and the resistance to the mass transfer in both the phases simultaneously. It has been shown that the application of metabolic reactions in the blood considerably intensify the concentration of diffusible components in the blood film.

N:0

PRACTICAL APPLICATION OF VEDIC KNOWLEDGE IN TODAY'S MODERN WORLD

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N:1

REMARKS ON CERTAIN APPROXIMATION TECHNIQUES OF GAṆITATILKA

V. Arora and V. Goel
Department of Mathematics and Statistics GKV, Hardwar 249 404

The main purpose of this paper is to discuss and compare techuniques for solving equations $x^2=a$ and $x^3=a$ (a is any positive integer) as provided in *Ganitatilaka* and elsewhere. Also we have extended this technique to non-perfect and decimal numbers.

N:2

A FEW IMPORTANT APPLICATIONS OF FIRST DEGREE INDETERMINATE ANALYSIS IN HINDU ASTRONOMY

S. L. Singh and N. B. Singh
Department of Mathematics and Statistics,
GKV and V.M.I.C., Hardwar

In this paper an attempt is made to discuss some applications of first degree Indeterminate Analysis in the field of Hindu Astronomy.

N:3

INDIAN MATHEMATICS : A BRIEF HISTORIC EVOLUTION WITH REFERENCE TO IRRATIONAL NUMBERS

V. Mishra and S. L. Singh

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This paper presents a brief historical account of chronology of Vedic literature, elementary treatment of surd numbers, theorem of square on the diagonal, square and cube-rooting, some aspects on varga prakrti, values of pi, and tangent-and pi-series, Datta's semi-geometrical approach for obtaining $\sqrt{2}$ is an ingenious technique based on the lines with the prescription of the sulba geometry. This method paves for evaluation of this and other surd numbers in various ways, thereby giving rise to a number of series representation for $\sqrt{N}$ ($N>0$, a positive integer).

N:4

ON THE HISTORY OF ANCIENT INDIAN MATHEMATICS

K. K. Velukutty

Department of Mathematics, College of Arts & Science, Coimbatore

Using 'Kalpaganitha' and 'Irainar' Akaporul', Kesari Balakrishna Pilla fixed the Sangham age as 498-817 AD. The Keralite legendary, Parachipettapanthrirukulam is analysed and the members were identified as the members of Navarathnas of Kulasekara I. This author, continuing the same stream of thought, identified Aryabhatta and Baskara I as Vararuchi and Pakkanar of Panthirukulam. Kesari himself identified Haridatta as Naranathuranthan. The year and place of birth of Aryabhatta are fixed as 581 (instead of 746) and Padaliputra in Tamil Nadu (instead of Padaliputra in Bihar). On the way of asserthing the hypothesos, the principale of three phases and addresses of scholars is introduced : Aryabhatta-Kumarilabhatta-Mochikeeranar, Haridatta-Naranathubranthan-Ponmudiyar and Baskara I-Pakkanar-Philantholmoosathu.

Concludingly, exposing the uncertainity of periods of Deergathamasu, Meghadhithi, Bharadwaja and Varahamihira, the needs for chronological sequence of events in Puranic (3102-967 BC) and Ancient (967 BC - 1169

AD) Indian Mathematics and the revival of finite mathematics due to Aryabhatta are stressed.

N:5

GEOMETRY AND ITS APPLICATIONS

Hukum Singh

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The notion of geometry originated in ancient times. They arose out of the necessity to determine the lengths, areas, and volumes of various objects. About 4000 yeas ago rules for determining areas and volumes were compiled in Egypt and Babylonia. Egypt people used this knowledge in measuring land, areas. The Greeks took over from the Egyptians and Babylonians their geometric findings about 2500 years ago. The Greek scholars discovered various geometric properties which were used for the further development of the different concepts.

Geometry in todays knowledge means the branch of mathematics dealing with spatial figures. In ancient Greece all of mathematics was regarded as geometry. B. Pascal spoke about geometry as mathematical way of thinking.

There are several branches of geometry viz., Euclidean geometry, Descarte's geometry, Non-Euclidean geometry, Gauss and Riemannian geometry etc. In this paper we have discussed about all these geometries and their applications in physics, chemistry and Biology.

N:6

APPLICATION OF INDETERMINATE ANALYSIS IN HINDU ASTRONOMY

Ramesh Chand

Department of Mathematics, V.M.I.C. Sec.5, BHEL, Hardwar

In this paper an attempt is made to explain the application of indeterminate Analysis for the solution of Hindu Astronomical Problems.

N:7

COMPUTATION OF ADHIKAMÃS BY CONTINUED FRACTION

Ramesh Chand

Department of Mathematics, V.M.I.C. Sec.5, BHEL, Hardwar

In this paper an attempt is made to explain the importance of Adhikamãs in Hindu Astronomy and computation of it by continued fraction.

N:8

GRAHALÃGHAVA AND COMPUTATION OF PLANETARY POSITION.

V. K. Sharma and Mustafizur Rahman

Department of Mathematics and Statistics, GKV, Hardwar-249404

In this paper an attempt is made to give the critical study of Grahalãghava and computation of planetary position, Tithi & Nakshatra etc, for any day.

N:9

A COMPARATIVE STUDY OF SIDEREAL REVOLUTIONS OF PLANETS IN A KALPA ACCORDING TO *BHÃSKARÃCHÃRYA* AND MODERN ASTRONOMY

V. K. Sharma and Mustafizur Rahman

Department of Mathematics and Statistics, GKV, Hardwar 249404

In this paper an attempt is made to give a comparative study of number of revolutions of planets in a Kalpa according to *Bhãskarãchãrya* and modern astronomy.

N:10

RELEVENCE OF VEDIC MATHEMATICS AND ITS APPLICATION IN GLOBAL PRESPECTIVE

G. P. Pokhariyal

Department of Mathematics, University of Nairobi, P O Box 30197, Nairobi, Kenya

Vedas are the most ancient compositions of knowledge containing the recollections of sages, seers and saints through observations and meditation in *Aranyas* of physical nature and the FORCE behind it; and must be viewed as the treasure of entire manking. Mathematics is the branch of knowledge which is mainly based on logic, reasoning and truth that has DIVINE input. *Vedic mathematics* provides a module for viewing things of the creation in a unique Vedic tradition where rules, theorems and propositions are presented in very concise and short way, through use of Sanskrit. Well established scientific theories and the current findings in science can also be considered to converge to the *Vedic* views ,in unfolding the mystery of nature.

N:11

SIMILARITY OF PLANE FIGURES : A REFLECTION IN INDIAN GEOMETRY

Vinod Mishra and S. L. Singh

Department of Mathematics, SLIET, Longowal 148106
Department of Mathematics, GKV, Hardwar 249404

This *talk* attempts to highlight Indian knowledge as regards to similarity concept and its implemetation to various geometrical problems.

N:12

MATHEMATICS IN ANCIENT INDIA

B. S. Rajput

Kumaun University, Nainital 263 001

The purpose of this *talk* is to present a brief account of significant achievements of ancient Indian savants to mathematical sciences.

N:13

VISVE-DEVA GODS-GROVR OF ANCIENT VEDIC PHILOSOPHY AND PHYSICAL DIMENSIONS OF MODERN SCIENTIFIC UNIVERSE

Rai Gyan Narain Prasad

Savitri Niwas (Behind Post Office) Mohaddipur Gorakhpur 273 008

In this paper we discuss dimensional relationship between the physical graduations and visvedevas.

N:14

BRIEF HISTORY OF DEVELOPMENT OF PROBABILITY

Vijayendra Kumar Sharma and Suresh Pal Kohli

Department of Mathematics and Statistics, GKV, Hardwar 249404

In this paper a brief account of development of probability theory through main contributions of Blaise Pascal, Jacoti Bermoulli, Abraham De Moivre, Baye, Gauss and Laplace etc is given.

N:15

VEDIC MATHEMATICS VERSUS MODERN MATHEMATICS WITH REGARD TO CERTAIN ASTRONOMICAL PHENOMENA

Shyama Charan Dubey

Vedic Research Institute, 51/102, Anand Nagar, Jalandhar 144 008

This *talk* intends to compare Vedic mathematics to with modern mathematics with reference to distance of sun from the earth, velocity of light, division of time and related issues.

N:16

MATHEMATICS (GAṆIT) - A VEDIC STUDY

Manu Dev Bandhu

Ved Department, Gurukula Kangri Univeristy, Hardwar 249 404

The founder of Arya Samaj Swami Dayanand Saraswati proclaimed- वेद सब सत्य विद्याओं का पुस्तक है। वेद का पढ़ना-पढ़ाना और सुनना-सुनाना सब आर्यों का परम धर्म है। The Vedas are sources of all true knowledge. To know the rule of mathematical measurement and their relation is quite essential. This science of calculations and compilations primarily falls under the heading of *Athervaveda.* The *trishaptah* (त्रिषप्ताः) means triple seven viz., twenty one. There is a specific upveda on this subject, "Arthveda". which deals with this science.

सहस्रशीर्षा पुरुषः सहस्राक्षः सहस्रपात्।
स भूमिं सर्वतः स्पृत्वाऽत्यतिष्ठद्दशाङ्गुलम्।।(Yaju 31/1)

In this verse सहस्र (Sahasra) means innumerable dashangulam (दशाङ्गुलम्) means the world which is made up of ten parts namely, ten gross elements, and 5 subtle elements. It also means that God is manifested on all the ten sides.

पश्येम शरदः शतं जीवेम शरदः शतम् (Yaju 36/24)

We may be able to see for hundred years, we may live long for hundred years.

कतम आदित्या इति द्वादश वै मासाः संवत्सरस्य एते आदित्याः
(Shatpath brahman 3/9/5)

Who are there Aditya ? These twelve months of the year are known twelve Aditya. the rays of the Sun have different effects on each month, on us.

एकं सत् Truth (God) is one. (Rigved)

N:17

FRACTIONAL CALCULUS : A PERSPICUITY OF ANALYSIS

P. K. Banerjee

Department of Mathematics, JNV University, Jodhpur 342 008

Knowing about the subject and the people who inspired since 1695, a houseful of great mathematicians, this is the story. By various amicable transformations, to day the subject of fractional calculus or differintegrals, has become an essential tool for applied mathematicians and for functional analysts. The text is prepared with a thought to introduce this branch of analysis to every one sitting in any corner of the globe : It contains basis, it has definitions and has various application sources. At this juncture of developments in mathematical analysis; I deem its presence extensively.

N:18

SUB-VEDA FOR CALCULATIONS

S. L. Singh

Department of Mathematics and Statistics, GKV, Hardwar 249 404

What is Vedic Mathematics ? Was there a separate *Veda* or *sub-Veda* for approximate and exact mathematical calculations ? This *talk* attempts to answer these questions.

N:19

NONLINEAR FLOW OF VEDIC MATHEMATICS

Virendra Arora

Department of Mathematics and Statistics, GKV, Hardwar 249 404

The intent of this *talk* is to discuss a few developmental aspects of mathematics of Vedic tradition.

N:20

REMARKS ON SOME FIRE ALTARS OF *S'ULBAS*

Virendra Arora and Nidhi Handa
Department of Mathematics and Statistics, GKV, Hardwar 249 404

The purpose of this note is to compare of few fire altars of *S'ulba* tradition.

N:21

CALCULATION OF THE AGE OF UNIVERSE -ANCIENT INDIAN MYTHOLOGICAL VIEW

K. N. Baramola and S. C. Sharma
Department of Physics, Govt. P.G. College, Kotdwara

Age of the universe has been a topic of wide interest for those who wish to compute it. Above all the age of the universe shows the discord amongst the various workers. In this paper we have compared the age of universe by *Hubble's Law* with that given in the *Kalpant* as given in Indian Mythology. Surprisingly and coincidently it comes out to be equal. In addition to this, other methods of dating of univese e.g. radio active dating, galactic evolution dating method, stellar evolution dating method etc. have also been discussed and compared with the age calculated in Indian mythology which is unparalled in other religions.

N:22

METHOD OF DIVISION OF SECOND DEGREE POLYNOMIALS IN TWO OR THREE VARIABLES BY FIRST DEGREE POLYNOMIAL IN TWO OR THREE VARIABLES

S. M. Chauthaiwale
Anglo Hindi Junior College, Yavatmal 445 001

The Horner's method of synthetic division for division of polynomials in one variable (say x) is well known with the fact that this method is restricted to linear divisors of the type $ax+b$, a and b are constants.

In the book named 'VEDIC MATHEMATICS' author Swami Bharati Krishna Tirthaji has explained methods of division of polynomials by *Paravartya Sutra* where the restriction on degree of divisors is removed and divisor may be any lesser degree polynomial than dividend. No doubt this is generalization of Hoener's Method of Synthetic division, and current Mathematicians should reconize this attempt as one of the best contribution to mathematics.

This paper further extends Swamiji's method, to division of second degree polynomials in two or three variables. In this method divisor is modified by Paravartya Sutra, the dividend is arrenged in descending order of degree of terms in any one variable followed by terms in another variable/s in similar manner. The division is carried out, with some necessary adjustments, similar to Swamiji's method.

N:23

MATHEMATICAL DATA IN VEDAS

(Padmashri) Kapil Dev Dwivedi
Vishvabharti Anusandhan Parisad, Gyanpur (Varanasi) 221 304

This *talk* intends to survey mathematical compositions and references in Vedas.

N:24

LARGE NUMBERS IN VEDAS

Dinesh Chandra Shastri
Department of Veda, GKV, Hardwar 249 404

This note reports incidence of certain natural numbers in Vedas.

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Journal of Natural & Physical Sciences Vol. 12 (1998), No. 1, 124-126

IMPORTANCE OF INFORMATION TECHNOLOGY IN EDUCATION IN MODERN TIMES

R. Anand*

Telecommunication via geostationary satellites and the worldwide network of computers have made the world a global village. Real time electronic communication between any two computers connected to the network, irrespective of their spatial separation, can be realised using the internet. The internet has shrunk the world in both space and time and made the concept of national frontiers as national as the boundary lines drawn on a globe.

In a village, the education of children in addition to bring the prerogative of the parents is a community responsibility, because children grow up together. Likewise in the present context, education of children who now live in a global village may have to be viewed with a broader perspective then as isolated national concerns. In other words we can call "Growing Up Together". This will require that our children study in a global classroom. Internet offers the possibility to realise teaching and learning in a "global classroom".

The School Education System in India

India being a vast country is full of diversity not only in its social and cultural but also in its educational system. It is therefore not surprising that the structure of education in the country became extremely heterogeneous as it evolved in the States in response to aspirations and contextual needs of the regional population. Therefore, the most important task for the Central and the State Governments, which work together in a federal set up, was to adopt a common structure of education. Because of recent efforts, the country has finally adopted a uniform structure of education. The structure has been split into three stages, comprising 10+2+3 years of studies.

Today, the world has changed. More and more jobs have become skilled. The youth will have to possess the skills required for initial employment and the ability to learn new skills to keep pace with the

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Ex. Professor, MIT (USA) and Managing Director, Altek Devices [P] Ltd., N. Delhi

changing nature of work. Another dimension to consider is that employment opportunities will be preferentially available to those who possess the required skills irrespective of their nationality. Therefore, young persons will require an education that can provide them better appreciation of life and culture in societies other than that of the society in which they grow up. The other crucial factor is the influence of information technology in nearly all occupations. Each person will have to learn how to use the computer as tool in the work that he/she may be doing and have the generic skills for using information technology effectively. Some of these skills are problem solving, critical thinking, decision making, accessing information, processing and dissemination of information after providing value addition to it.

The Students and the Teachers can take advantage of the vast educational resources available in the countries of the region, elsewhere in the world through the internet and other communication technologies. The personality of a child is shaped in a cultural context. By enabling young persons to grow up together in an interactive environment with peers in different countries will help them to attain comparable intellectual heights and develop international understanding. To make this happen we require participation of appropriate organisations in the region for working.

It may be pointed out that Educational Institutions in India emphasis is given on teaching of traditional subjects such as languages, mathematics, science and social sciences. What is required is that alongwith traditional subjects. They should be able to learn decision making skills, communication skills and interpersonal skills. These can be achieved by exposing students to real life problem, involving for their solution a collaborative effort. Implementation of such an approach will require that Educational Institutions, irrespective of their physical separation, arrange for their students to interact with one another to enable them to work on problems that can be best solved collectively. The internet provides an effective mode for enabling interaction in spite of physical separation. In addition, the global networking of computers has made if possible to shere educational resources whichout encumbrance and that too at near zero cost. Individuals and institutions are putting up information on every conceivable area of human interest on the Internet. The Internet is an ideal medium for interactive self-learning and if used intelligently with develop learning to learn. Therefore, the computer and the network to which it may be connected have to be used as tools for achieving the stated objectives and not as subjects of study.

It is expected that these competencies will the information technology will develop in them the confidence to upgrade their skills, as they would have acquired the ability of learning to learn.

The topics on which the Indian students should be able to carry out in-depth study and acquire information that they can share over the internet with their international peers, say students in Australia, are those that they have inherited, grown up with or have easy access to. For them, it would be a voyage into their own heritage and culture for discovering India. if similar approach is adopted by all the participating countries the children with their knowledge, understanding and experience will be able to mutually benefit and develop a realistic appreciation of each other's culture.

Role of Educational Institution

The Educational Institutions should take the responsibility of :

(a) Development of modules for students for acquiring the planned competencies
(b) Training of teachers
(c) Networking with professionals for providing expert support
(d) Obtaining feedback from students
(e) Making available on the internet its resources
(f) Answering students queries by e-mail

Looking ahead

It is accepted that a majority of schools in India have inadequate infrastructure and it may not be possible for the State to provide computers in sufficient numbers to each Educational Institutions for implementing the above programme. If with the pilot project, it can be convincingly shown that children who are information technology literates have an edge in an information society, the community may come forward to meet the needed support.

Papers/Talks : Late Arrivals

T. T. Raghunathan : Enumeration of Bipartite Self Complementary Graphs.

M. M. Shikare : Splitting operation and Minors of Binary Matroids.

N. S. Bhave : Generalised Bipartite Complement of a Graph.

Y. S. Pawar : 0-1 Distributive Lattices and Complementedness.

S. K. Nimbhorkar : Theory of Rings and Lattices.

M. P. Wasadikar : Atomistic Semi-lattices.

Pornima Pawar : Approximation by Data Reduction.

M. A. Pathan : Lie Theoretic Approach to generalised Bessels functions.

P. Bhattacharya : Comlex dynamics during last 25 years.

Sundar Lal : Some Secret sharing schemes.

S. M. Bhave : Set Theory in Ancient Indian Logic.

V. K. Grutu : Historic Marginal Note and Fermat's Last Theorem.

K. N. Joshi : Realization of complex angular momentum operator.

64th Indian Mathematical Society Conference
and
Symposia
on
Traditions of Vedic Mathematics and Applications
&
Fixed Point Theory in Nonlinear Analysis & Applications

December 19-22, 1998
Gurukula Kangri Vishwavidyalaya, Hardwar 249 404

DAILY PROGRAMME

DECEMBER 19, 1998

8.30 - 9.30	REGISTRATION
9.45 - 11.30	INAUGURAL SESSION Flag Hoisting & Procession Invocation Welcome Address by the Vice-Chancellor, GKV Report by the General Secretary Report by the Administrative Secretary Report by the Academic Secretary Inaugural Address by the Chief Guest Address by the Guest of Honour Presidential Address (General) by Prof. B. K. Lahiri Release of "Resonance of Ramajujan's Mathematics Vol. III : R. P. Agarwal": by the Chief Guest; Presentation by the General Secretary Award of P. L. Bhatnagar Memorial Prize to Shri Abhinav Kumar GKV Honours Mathematicians; Presentation by the Principal Vote of Thanks by the Local Secretary National Anthem
11.30 - 12.00	TEA
12.00 - 13.00	PRESIDENTIAL ADDRESS (Technical) : B. K. Lahiri Chairperson : R. P. Agarwal
13.00 - 14.30	GROUP PHOTOGRAPH FOLLOWED BY LUNCH
14.30 - 15.45	NINTH HANSRAJ GUPTA MEMORIAL AWARD LECTURE T. Parthasarathy : An Old Problem of Samuelson and the Jacobian Conjecture Chairperson : A. R. Singal

15.45 - 16.00 TEA
16.00 - 18.00 SPECIAL PAPER READING SESSION :
Presentation of papers for the Award of Prizes
18.00 VISIT TO JAIRAM ASHRAM AND DINNER

DECEMBER 20, 1998

09.30-10.00 INVITED TALKS :
A.R. Singal : The Millenium Bug (The *Y2K* Problem)
Chairperson : T. Parthasarathy
10.00-11.15 NINTH V. RAMASWAMY AIYER MEMORIAL AWARD LECTURE
V.M. Shah : A Peep Into the Grass Roots of All Mathematics Education
Chairperson : M.K. Singal
11.15 - 11.30 TEA

11.30 - 13.00 **SESSION - I**
SESSION ON LATTICE THEORY
Co-ordiantior : N.K. Thakare
Speakers : Y.S. Pawar
M.P. Wasadikar
S.K. Nimbhorkar

11.30-13.00 **SESSION - II**
FIXED POINT THEORY SYMPOSIUM SESSION
Co-ordinators : B. K. Lahiri and S. L. Singh
Inaugural Speaker : B.S. Yadav
Other Speakers : P.P. Murthy, R.N. Mukherjaee,
R.P. Pant, Geetha S Rao and others.

13.00 -14.00 LUNCH AT GURUKULA KANGRI PHARMACY
14.00 EXCURSION TO RISHIKESH INCLUDING VISIT TO PRAMARTH NIKETAN
17.00 - 18.30 SHRI GANGA ARATI, Blessings by Svami Chidanand Muni Ji and Prasadam

DECEMBER 21, 1998

09.30 - 10.00 INVITED TALKS :
S.B. Nimse : Topological Quantum Field Theory
Chairperson : V.M. Shah

10.00 - 11.15 TWELTH P.L. BHATNAGAR MEMORIAL AWARD LECTURE
S.K. Malik : Patterns and Chaos in Fluids
Chairperson : H. C. Khare

11.15-11.30 TEA
11.30-13.00 INVITED TALKS

SESSION - I
Sudhir Ghorpade, C. M. Joshi, P. K. Banerjee
Chairperson : N. K. Thakare

SESSION - II
D. V. Pai, Poornima Pawar, V. K. Gurtu, R. Anand
Chairperson : Hukum Singh

SESSION - III
S. M. Bhave, M. A. Pathan, P. Bhattacharya,
Chairperson : A. K. Agarwal

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14.00 - 15.00 PAPER PRESENTATION :
Session I : Chairperson : M. A. Pathan
Session II : Chairperson : S. B. Nimse
Session III : Chairperson : Manjul Gupta
Session IV : Chairperson : L. Radhakrishna
Session V : Chairperson : T. T. Raghunathan

15.00 - 16.45 **SESSION - I**
VEDIC MATHEMATICS SYMPOSIUM SESSION
Co-ordinators : S. L. Singh & Virendra Arora

Inaugural Speaker : Pranav Pandya
Other Speakers : B. S. Rajput, G. P. Pokhariyal, S. R. Pant, P. K. Mukherjee, Vinod Mishra, S. C. Dubey and others.

15.00 - 16.45 **SESSION - II**
SESSION ON GRAPH THEORY
Co-ordinator : T. T. Raghunathan
Speakers : M. M. Shikare, N. S. Bhave

16.45 - 17.00 TEA
17.00 - 18.00 GENERAL BODY MEETING
18.00 VISIT TO SHANTIKUNJ AND DINNER

DECEMBER 22, 1998

09.30 - 09.45 UNVEILING OF PORTRAIT OF RAMANUJAN
09.45 - 10.15 R. P. Agarwal : Ramanujan's Mathematics In India
10.15 - 11.15 A. K. Agarwal : Ninth Ramanujan Memorial Award Lecture "The legacy of Ramanujan".

11.15 - 12.30 TEA

11.30 - 12.30 **SESSION - I**
INVITED TALKS :
A. Verma :
R. Y. Denis : Ramanujan's Continued Fractions
S. N. Singh : Ramanujan's Mock Theta Function
Concluding remarks : R. P. Agarwal

11.30 - 12.30 **SESSION - II**
PAPER PRESENTATION :
SESSION VI : Chairperson : C. M. Joshi
SESSION VII : Chairperson : Sunder Lal
SESSION VIII : Chairperson : P. K. Banerjee
SESSION IX : Chairperson : Sudhir Ghorpade

12.30 - 01.00 VALEDICTORY REMARKS BY THE PRESIDENT IMS
01.00 VISIT TO NIRDHAN NIKETAN AND LUNCH

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